IV. Approximation Algorithms: Covering Problems

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UNIVERSITY OF CAMBRIDGE

Outline

Introduction

Vertex Cover

Motivation

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2. Isolate important special cases which can be solved in polynomial-time.
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We will call these **approximation algorithms**.

Performance Ratios for Approximation Algorithms

Approximation Ratio

An algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the cost C of the returned solution and optimal cost C^* satisfy:

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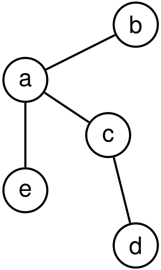
- Goal: Find a minimum-cardinality subset $V' \subseteq V$ such that if $(u, v) \in E(G)$, then $u \in V'$ or $v \in V'$.



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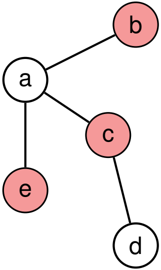

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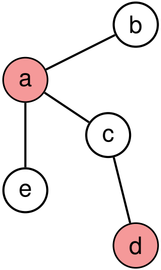

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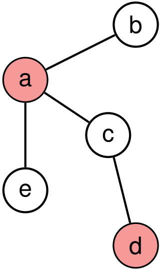

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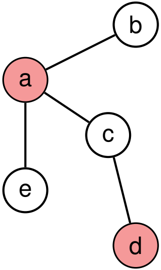

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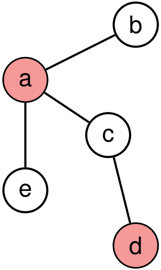

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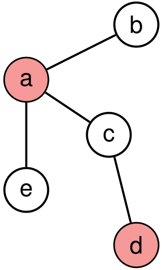

Applications:

- Every edge forms a task, and every vertex represents a person/machine which can execute that task

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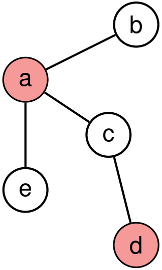

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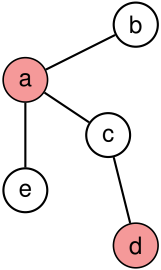

Applications:

- Every edge forms a task, and every vertex represents a person/machine which can execute that task
- Perform all tasks with the minimal amount of resources
- Extensions: weighted vertices or hypergraphs ($\rightsquigarrow$ Set-Covering Problem)

An Approximation Algorithm based on Greedy

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APPROX-VERTEX-COVER(G)
C = ∅
E' = G.E
while E' ≠ ∅
    let (u, v) be an arbitrary edge of E'
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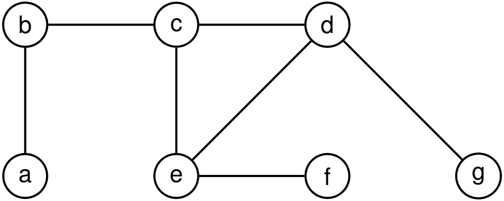

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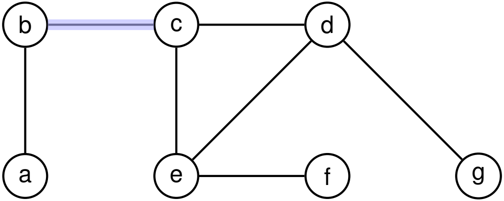

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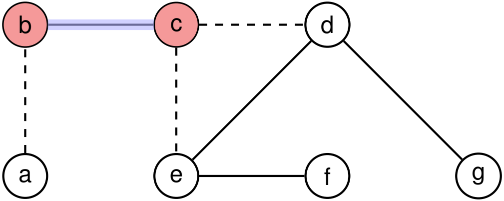

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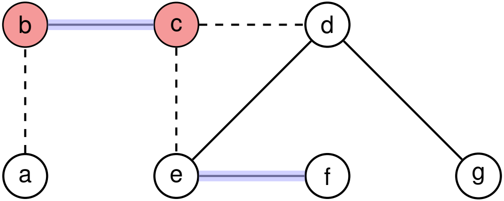

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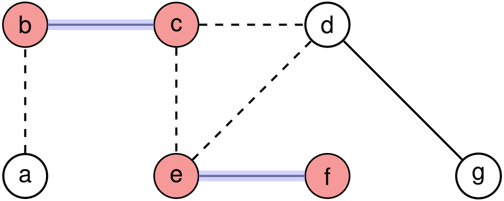

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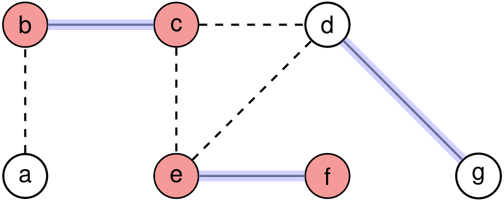

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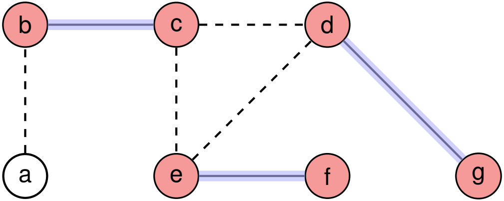

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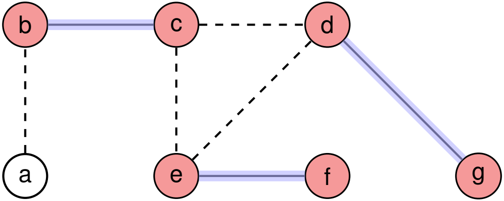

APPROX-VERTEX-COVER produces a set of size 6.

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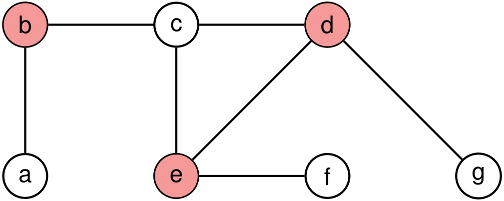

The optimal solution has size 3.

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4      let (u, v) be an arbitrary edge of E'
5      C = C ∪ {u, v}
6      remove from E' every edge incident on either u or v
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Theorem 35.1

APPROX-VERTEX-COVER is a poly-time 2-approximation algorithm.

Proof:

- Running time is $O(V+E)$ (using adjacency lists to represent E')
- Let $A \subseteq E$ denote the set of edges picked in line 4
- Every optimal cover C^* must include at least one endpoint of edges in A, and edges in A do not share a common endpoint: $\boxed{|C^*| \geq |A|}$
- Every edge in A contributes 2 vertices to $|C|$: $\boxed{|C| = 2|A| \leq 2|C^*|.}$

Analysis of Greedy for Vertex Cover

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Analysis of Greedy for Vertex Cover

APPROX-VERTEX-COVER(G)

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3 **while** $E' \neq \emptyset$
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A "vertex-based" Greedy that adds **one** vertex at each iteration fails to achieve an approximation ratio of 2 (Exercise)!

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Solving Special Cases

Strategies to cope with NP-complete problems

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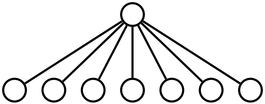

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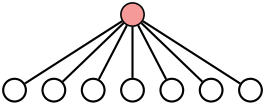

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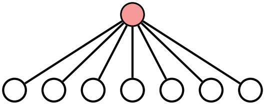

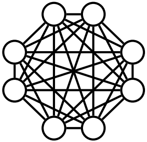

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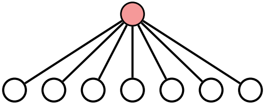

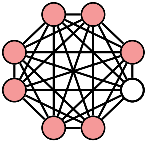

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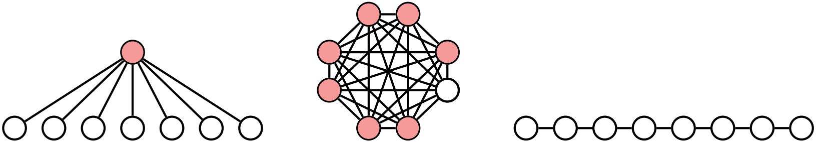

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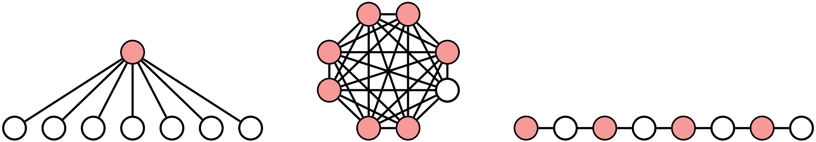
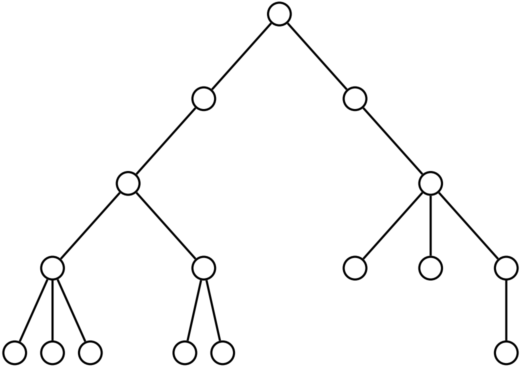


Vertex Cover on Trees



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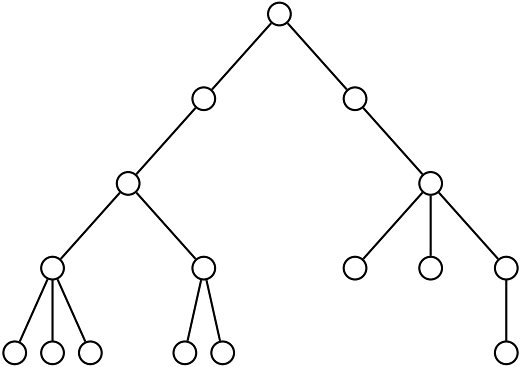

There exists an optimal vertex cover which does not include any leaves.

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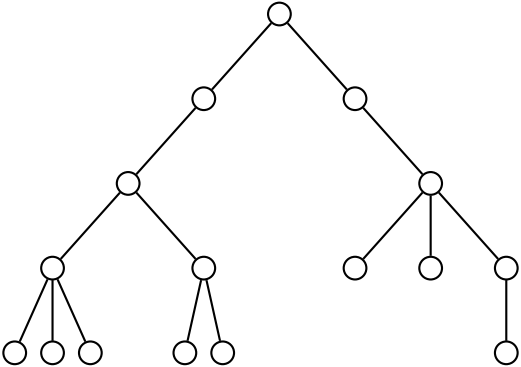

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Exchange-Argument: Replace any leaf in the cover by its parent.

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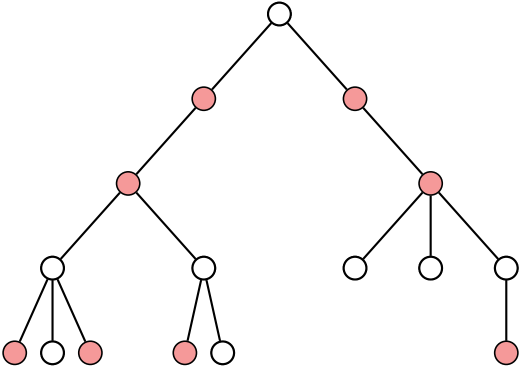

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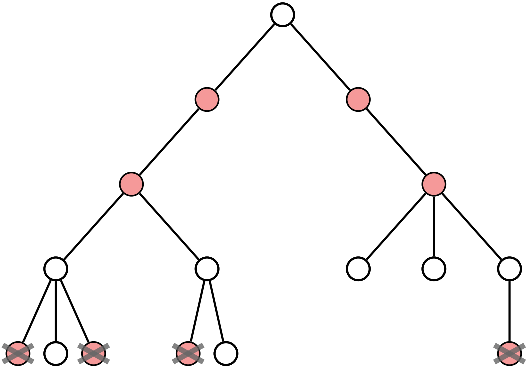

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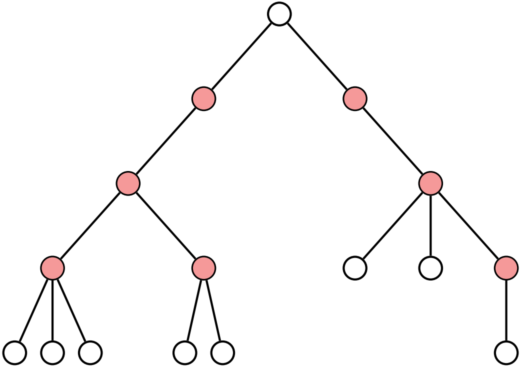

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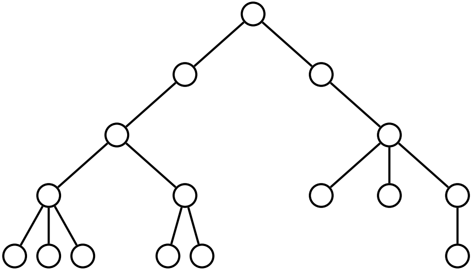
Solution is also optimal. (Use inductively the existence of an optimal vertex cover without leaves)

Execution on a Small Example



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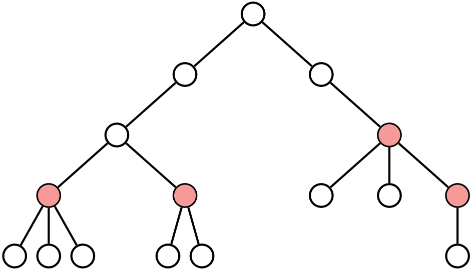

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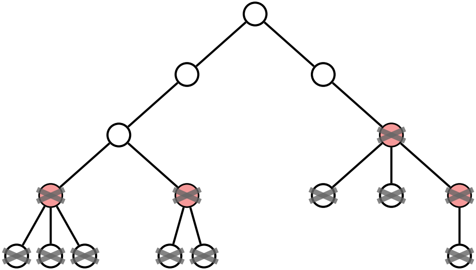

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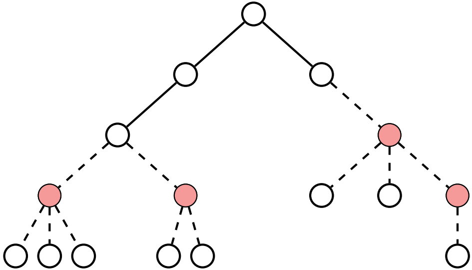

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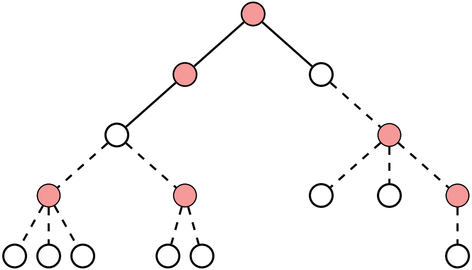

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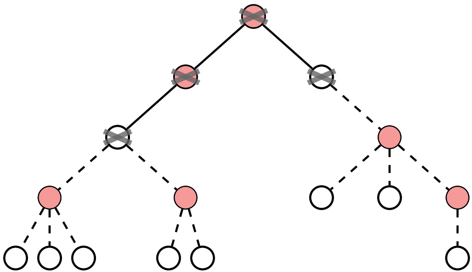

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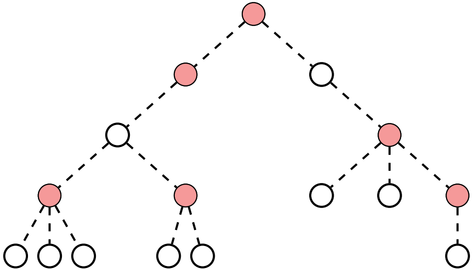

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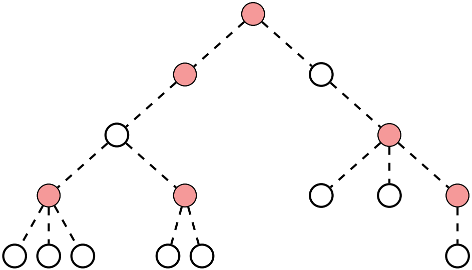

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Problem can be also solved on bipartite graphs, using Max-Flows and Min-Cuts.

Exact Algorithms

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Simple Brute-Force Search would take $\approx \binom{n}{k} = \Theta(n^k)$ time.

Towards a more efficient Search

Substructure Lemma

Consider a graph $G = (V, E)$, edge $\{u, v\} \in E(G)$ and integer $k \geq 1$. Let G_u be the graph obtained by deleting u and its incident edges (G_v is defined similarly). Then G has a vertex cover of size k if and only if G_u or G_v (or both) have a vertex cover of size $k - 1$.

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Reminiscent of Dynamic Programming.

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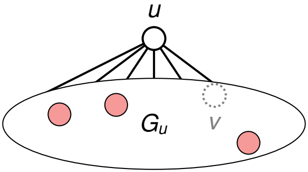

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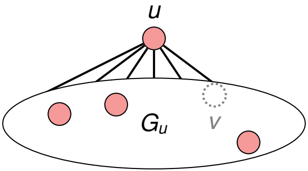
Adding u yields a vertex cover of G which is of size k



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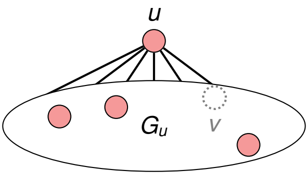

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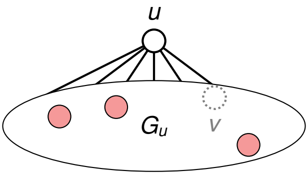
Removing u from C yields a vertex cover of G_u which is of size $k-1$. $\square$



A More Efficient Search Algorithm

VERTEX-COVER-SEARCH(G, k)

If $E = \emptyset$ **return** $\emptyset$
If $k = 0$ and $E \neq \emptyset$ **return** $\perp$
Pick an arbitrary edge $(u, v) \in E$
$S_1 =$ VERTEX-COVER-SEARCH$(G_u, k-1)$
$S_2 =$ VERTEX-COVER-SEARCH$(G_v, k-1)$
if $S_1 \neq \perp$ **return** $S_1 \cup \{u\}$
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Correctness follows by the Substructure Lemma and induction.

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Running time:

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Running time:

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- $O(E)$ work per recursive call

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If $k = 0$ and $E \neq \emptyset$ **return** $\perp$
Pick an arbitrary edge $(u, v) \in E$
$S_1 =$ VERTEX-COVER-SEARCH$(G_u, k - 1)$
$S_2 =$ VERTEX-COVER-SEARCH$(G_v, k - 1)$
if $S_1 \neq \perp$ **return** $S_1 \cup \{u\}$
if $S_2 \neq \perp$ **return** $S_2 \cup \{v\}$
return $\perp$

Running time:

- Depth k, branching factor 2 $\Rightarrow$ total number of calls is $O(2^k)$
- $O(E)$ work per recursive call
- Total runtime: $O(2^k \cdot E)$.

A More Efficient Search Algorithm

$\textsc{Vertex-Cover-Search}(G, k)$

If $E = \emptyset$ **return** $\emptyset$
If $k = 0$ and $E \neq \emptyset$ **return** $\perp$
Pick an arbitrary edge $(u, v) \in E$
$S_1 = \textsc{Vertex-Cover-Search}(G_u, k-1)$
$S_2 = \textsc{Vertex-Cover-Search}(G_v, k-1)$
if $S_1 \neq \perp$ **return** $S_1 \cup \{u\}$
if $S_2 \neq \perp$ **return** $S_2 \cup \{v\}$
return $\perp$

Running time:

- Depth k, branching factor 2 $\Rightarrow$ total number of calls is $O(2^k)$
- $O(E)$ work per recursive call
- Total runtime: $O(2^k \cdot E)$.

exponential in k, but much better than $\Theta(n^k)$ (i.e., still polynomial for $k = O(\log n)$)