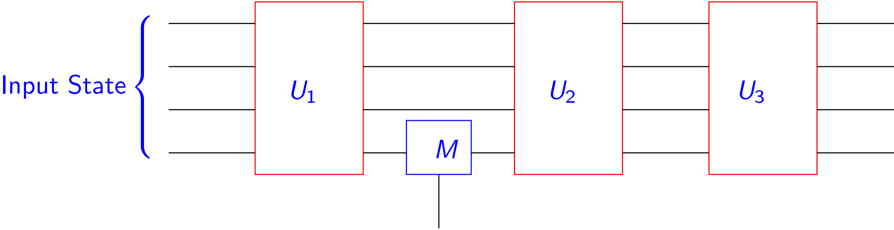
Quantum Computing
Lecture 4

Models of Quantum Computation

Anuj Dawar

Quantum Circuits

A *quantum circuit* is a sequence of unitary operations and measurements on an n-qubit state.



Note: each U_i is described by a $2^n \times 2^n$ matrix.

Algorithms

A *quantum algorithm* specifies, for each n, a sequence

$$\mathcal{O}_n = O_1 \ldots O_k$$

of n-qubit operations.

The map $n \to \mathcal{O}_n$ must be computable.

i.e. the individual circuits must be generated from a common pattern.

All measurements can be deferred to the end (possibly, at the expense of increasing the number of qubits).

Model of Computation

As a model of computation, this is parasitic on classical models.

what is computable is not independently determined

Purely quantum models can be defined. We will see more on this in Lecture 8.

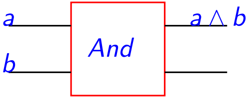
What computations can be performed in the model as defined?

What functions can be computed?
What decision problems are decidable?

Can all such computations be performed with some fixed set of unitary operations?

Simulating Boolean Gates

Could we find a quantum circuit to simulate a classical *And* gate?



This would require $\mathit{And}: \quad |00\rangle \mapsto |0x\rangle, \quad |01\rangle \mapsto |0y\rangle$
$\qquad\qquad\qquad\qquad |10\rangle \mapsto |0z\rangle, \quad |11\rangle \mapsto |1w\rangle$

There is no *unitary* operation of this form.
Unitary operations are reversible. No information can be lost in the process.

Computing a Function

If $f : \{0,1\}^n \to \{0,1\}^m$ is a *Boolean function*, the map

$$|x\rangle \mapsto |f(x)\rangle$$

may not be unitary.

We will, instead seek to implement

$$|x\rangle \otimes |0\rangle \mapsto |x\rangle \otimes |f(x)\rangle$$

Exercise: Describe a unitary operation that implements the Boolean *And* in this sense.

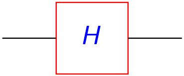
One-Qubit Gates

We have already seen the *Pauli Gates*:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

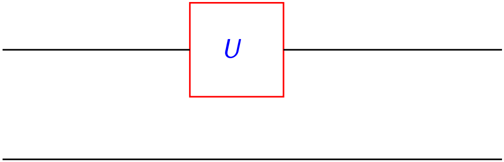
Another useful *one-qubit* gate is the *Hadamard gate*:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



Gates on a Multi-Qubit State

When we draw a circuit with a one-qubit gate, this must be read as a unitary operation on the *entire state*.

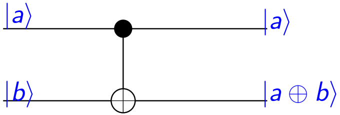


$$U \otimes I$$

This does not change measurement outcomes on the second qubit.

Controlled Not

The *Controlled Not* is a 2-qubit gate:

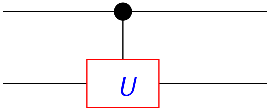


The controlled not flips the second qubit if the first qubit is $|1\rangle$ and leaves it unchanged if it's $|0\rangle$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

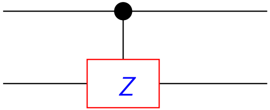
Controlled U

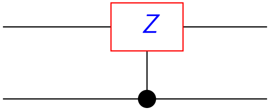
More generally, we can define, for any single qubit operation U, the *Controlled U* gate:



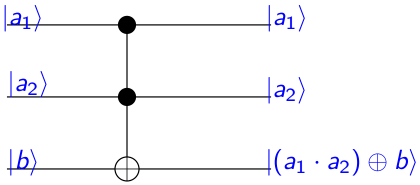
$$
\begin{aligned}
|0x\rangle &\mapsto |0x\rangle \\
|1x\rangle &\mapsto |1, Ux\rangle
\end{aligned}
$$

Particularly useful is the controlled-Z gate:





Toffoli Gate



The *Toffoli Gate* is a 3-qubit gate.
It has a classical counterpart which can be used to simulate standard Boolean operations

A *permutation matrix* is a unitary matrix where all entries are 0 or 1.

Any $2^n \times 2^n$ permutation matrix can be implemented using only Toffoli gates.

Classical Reversible Computation

A Boolean function $f : \{0,1\}^n \to \{0,1\}^n$ is *reversible* if it's described by a $2^n \times 2^n$ permutation matrix.

For any function $g : \{0,1\}^n \to \{0,1\}^m$, there is a reversible function $g' : \{0,1\}^{m+n} \to \{0,1\}^{m+n}$ with

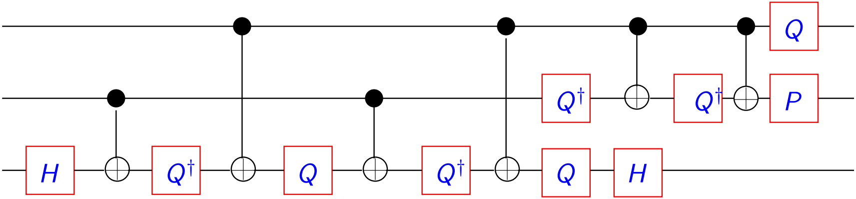
$$g'(x,0) = (x, g(x)).$$

Toffoli gates are *universal* for reversible computation.

The Toffoli gate cannot be implemented using 2-bit reversible classical gates.

Quantum Toffoli Gate

The Toffoli gate can be implemented using 2-qubit quantum gates.



where, $P = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$.

Universal Set of Gates

Fact: Any unitary operation on n qubits can be implemented by a sequence of 2-qubit operations.

Fact: Any unitary operation can be implemented by a combination of C-NOTs and single qubit operations.

Fact: Any unitary operation can be *approximated* to any required degree of accuracy using only C-NOTs, H, P and Q.

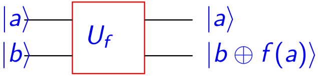
These can serve as our finite set of gates for quantum computation.

Deutsch-Jozsa Problem

Given a function $f : \{0,1\} \to \{0,1\}$, determine whether f is constant or balanced.

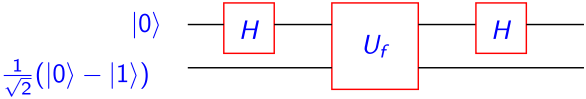
Classically, this requires *two* calls to the function f.

But, if we are given the *quantum black box*:



One use of the box suffices

Deutsch-Jozsa Algorithm



U_f with input $|x\rangle$ and $|0\rangle - |1\rangle$ is just a phase shift.
It changes phase by $(-1)^{f(x)}$.
When $|x\rangle = H|0\rangle$, this gives $(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle$.

Final result is $[(-1)^{f(0)} + (-1)^{f(1)}]|0\rangle + [(-1)^{f(0)} - (-1)^{f(1)}]|1\rangle$
which is $|0\rangle$ if f is constant and $|1\rangle$ if f is balanced.