L11: Algebraic Path Problems with applications to Internet Routing

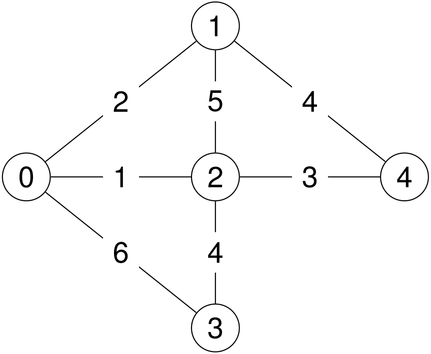
Lectures 05 — 07

Timothy G. Griffin

timothy.griffin@cl.cam.ac.uk
Computer Laboratory
University of Cambridge, UK

Michaelmas Term, 2014

Shortest paths example, $(\mathbb{N}^{\infty}, \min, +)$



The adjacency matrix

$$
\mathbf{A} \quad = \quad
\begin{array}{c}
 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\left[\begin{array}{ccccc}
\infty & 2 & 1 & 6 & \infty \\
2 & \infty & 5 & \infty & 4 \\
1 & 5 & \infty & 4 & 3 \\
6 & \infty & 4 & \infty & \infty \\
\infty & 4 & 3 & \infty & \infty
\end{array}\right]
\end{array}
$$

Note that the longest shortest path is $(1, 0, 2, 3)$ of length 3 and weight 7.

$(\min, +)$ example

Our theorem tells us that $\mathbf{A}^* = \mathbf{A}^{(n-1)} = \mathbf{A}^{(4)}$

$$\mathbf{A}^* = \mathbf{A}^{(4)} = \mathbf{I} \ \min \ \mathbf{A} \ \min \ \mathbf{A}^2 \ \min \ \mathbf{A}^3 \ \min \ \mathbf{A}^4 = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \left[\begin{array}{ccccc} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{array}\right] \end{array}$$

$(\min, +)$ example

$$\mathbf{A} = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \infty & \underline{2} & \underline{1} & 6 & \infty \\ \underline{2} & \infty & 5 & \infty & \underline{4} \\ \underline{1} & 5 & \infty & \underline{4} & \underline{3} \\ 6 & \infty & \underline{4} & \infty & \infty \\ \infty & \underline{4} & \underline{3} & \infty & \infty \end{bmatrix} \end{array}$$

$$\mathbf{A}^3 = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 8 & 4 & 3 & 8 & 10 \\ 4 & 8 & 7 & \underline{7} & 6 \\ 3 & 7 & 8 & 6 & 5 \\ 8 & \underline{7} & 6 & 11 & 10 \\ 10 & 6 & 5 & 10 & 12 \end{bmatrix} \end{array}$$

$$\mathbf{A}^2 = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 2 & 6 & 7 & \underline{5} & \underline{4} \\ 6 & 4 & \underline{3} & 8 & 8 \\ 7 & \underline{3} & 2 & 7 & 9 \\ \underline{5} & 8 & 7 & 8 & \underline{7} \\ \underline{4} & 8 & 9 & \underline{7} & 6 \end{bmatrix} \end{array}$$

$$\mathbf{A}^4 = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 4 & 8 & 9 & 7 & 6 \\ 8 & 6 & 5 & 10 & 10 \\ 9 & 5 & 4 & 9 & 11 \\ 7 & 10 & 9 & 10 & 9 \\ 6 & 10 & 11 & 9 & 8 \end{bmatrix} \end{array}$$

First appearance of final value is in red and underlined. Remember: we are looking at all paths of a given length, even those with cycles!

A “better” way — our basic algorithm

$$
\begin{aligned}
\mathbf{A}^{\langle 0 \rangle} &= \mathbf{I} \\
\mathbf{A}^{\langle k+1 \rangle} &= \mathbf{A}\mathbf{A}^{\langle k \rangle} \oplus \mathbf{I}
\end{aligned}
$$

Lemma

$$\mathbf{A}^{\langle k \rangle} = \mathbf{A}^{(k)} = \mathbf{I} \oplus \mathbf{A}^{1} \oplus \mathbf{A}^{2} \oplus \cdots \oplus \mathbf{A}^{k}$$

back to $(\min, +)$ example

$$\mathbf{A}^{\langle 1 \rangle} = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 2 & 1 & 6 & \infty \\ 2 & 0 & 5 & \infty & 4 \\ 1 & 5 & 0 & 4 & 3 \\ 6 & \infty & 4 & 0 & \infty \\ \infty & 4 & 3 & \infty & 0 \end{bmatrix} \end{array}$$

$$\mathbf{A}^{\langle 3 \rangle} = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{A}^{\langle 2 \rangle} = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 8 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 8 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{bmatrix} \end{array}$$

A note on $\mathbf{A}$ vs. $\mathbf{A} \oplus \mathbf{I}$

Lemma 6.0

If $\oplus$ is idempotent, then

$$(\mathbf{A} \oplus \mathbf{I})^k = \mathbf{A}^{(k)}.$$

Proof. Base case: When $k = 0$ both expressions are $\mathbf{I}$.
Assume $(\mathbf{A} \oplus \mathbf{I})^k = \mathbf{A}^{(k)}$. Then

$$\begin{aligned}
(\mathbf{A} \oplus \mathbf{I})^{k+1} &= (\mathbf{A} \oplus \mathbf{I})(\mathbf{A} \oplus \mathbf{I})^k \\
&= (\mathbf{A} \oplus \mathbf{I})\mathbf{A}^{(k)} \\
&= \mathbf{A}\mathbf{A}^{(k)} \oplus \mathbf{A}^{(k)} \\
&= \mathbf{A}(\mathbf{I} \oplus \mathbf{A} \oplus \cdots \oplus \mathbf{A}^k) \oplus \mathbf{A}^{(k)} \\
&= \mathbf{A} \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^{k+1} \oplus \mathbf{A}^{(k)} \\
&= \mathbf{A}^{k+1} \oplus \mathbf{A}^{(k)} \\
&= \mathbf{A}^{(k+1)}
\end{aligned}$$

Solving (some) equations

Theorem 6.1

If $\mathbf{A}$ is q-stable, then $\mathbf{A}^*$ solves the equations

$$\mathbf{L} = \mathbf{A}\mathbf{L} \oplus \mathbf{I}$$

and

$$\mathbf{R} = \mathbf{R}\mathbf{A} \oplus \mathbf{I}.$$

For example, to show $\mathbf{L} = \mathbf{A}^*$ solves the first equation:

$$\begin{array}{rcl}
\mathbf{A}^* & = & \mathbf{A}^{(q)} \\
& = & \mathbf{A}^{(q+1)} \\
& = & \mathbf{A}^{q+1} \oplus \mathbf{A}^{q} \oplus \ldots \oplus \mathbf{A}^{2} \oplus \mathbf{A} \oplus \mathbf{I} \\
& = & \mathbf{A}(\mathbf{A}^{q} \oplus \mathbf{A}^{q-1} \oplus \ldots \oplus \mathbf{A} \oplus \mathbf{I}) \oplus \mathbf{I} \\
& = & \mathbf{A}\mathbf{A}^{(q)} \oplus \mathbf{I} \\
& = & \mathbf{A}\mathbf{A}^{*} \oplus \mathbf{I}
\end{array}$$

Note that if we replace the assumption "**A** is q-stable" with "$\mathbf{A}^*$ exists," then we require that $\otimes$ distributes over infinite sums.

A more general result

Theorem Left-Right

If $\mathbf{A}$ is q-stable, then $\mathbf{L} = \mathbf{A}^*\mathbf{B}$ solves the equation

$$\mathbf{L} = \mathbf{A}\mathbf{L} \oplus \mathbf{B}$$

and $\mathbf{R} = \mathbf{B}\mathbf{A}^*$ solves

$$\mathbf{R} = \mathbf{R}\mathbf{A} \oplus \mathbf{B}.$$

For the first equation:

$$\begin{aligned}
\mathbf{A}^*\mathbf{B} &= \mathbf{A}^{(q)}\mathbf{B} \\
&= \mathbf{A}^{(q+1)}\mathbf{B} \\
&= (\mathbf{A}^{q+1} \oplus \mathbf{A}^{q} \oplus \ldots \oplus \mathbf{A}^{2} \oplus \mathbf{A} \oplus \mathbf{I})\mathbf{B} \\
&= (\mathbf{A}^{q+1} \oplus \mathbf{A}^{q} \oplus \ldots \oplus \mathbf{A}^{2} \oplus \mathbf{A})\mathbf{B} \oplus \mathbf{B} \\
&= \mathbf{A}(\mathbf{A}^{q} \oplus \mathbf{A}^{q-1} \oplus \ldots \oplus \mathbf{A} \oplus \mathbf{I})\mathbf{B} \oplus \mathbf{B} \\
&= \mathbf{A}(\mathbf{A}^{(q)}\mathbf{B}) \oplus \mathbf{B} \\
&= \mathbf{A}(\mathbf{A}^*\mathbf{B}) \oplus \mathbf{B}
\end{aligned}$$

Use Theorem Left-Right to Work this out

Theorem (John Conway, 1971)

If

$$\mathbf{A} = \left(\begin{array}{c|c} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \hline \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{array}\right)$$

then $\mathbf{A}^*$ can be written as

$$\left(\begin{array}{c|c} (\mathbf{A}_{1,1} \oplus \mathbf{A}_{1,2}\mathbf{A}_{2,2}^*\mathbf{A}_{2,1})^* & \mathbf{A}_{1,1}^*\mathbf{A}_{1,2}(\mathbf{A}_{2,2} \oplus \mathbf{A}_{2,1}\mathbf{A}_{1,1}^*\mathbf{A}_{1,2})^* \\ \hline \mathbf{A}_{2,2}^*\mathbf{A}_{2,1}(\mathbf{A}_{1,1} \oplus \mathbf{A}_{1,2}\mathbf{A}_{2,2}^*\mathbf{A}_{2,1})^* & (\mathbf{A}_{2,2} \oplus \mathbf{A}_{2,1}\mathbf{A}_{1,1}^*\mathbf{A}_{1,2})^* \end{array}\right)$$

The “best” solution

Suppose $\mathbf{Y}$ is a matrix such that

$$\mathbf{Y} = \mathbf{AY} \oplus \mathbf{I}$$

$$\begin{array}{rcl}
\mathbf{Y} & = & \mathbf{AY} \oplus \mathbf{I} \\
 & = & \mathbf{A}^1\mathbf{Y} \oplus \mathbf{A}^{(0)} \\
 & = & \mathbf{A}((\mathbf{AY} \oplus \mathbf{I})) \oplus \mathbf{I} \\
 & = & \mathbf{A}^2\mathbf{Y} \oplus \mathbf{A} \oplus \mathbf{I} \\
 & = & \mathbf{A}^2\mathbf{Y} \oplus \mathbf{A}^{(1)} \\
 \vdots & & \vdots \quad \vdots \\
 & = & \mathbf{A}^{k+1}\mathbf{Y} \oplus \mathbf{A}^{(k)}
\end{array}$$

If $\mathbf{A}$ is q-stable and $q < k$, then

$$\mathbf{Y} = \mathbf{A}^k\mathbf{Y} \oplus \mathbf{A}^*$$

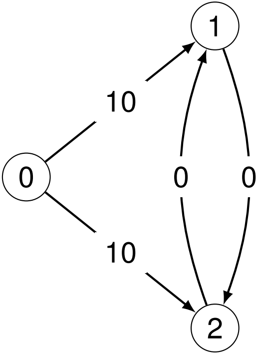
$$\mathbf{Y} \trianglelefteq^L_\oplus \mathbf{A}^*$$

and if $\oplus$ is idempotent, then

$$\mathbf{Y} \leq^L_\oplus \mathbf{A}^*$$

So $\mathbf{A}^*$ is the largest solution. What does this mean in terms of the sp semiring?

Example with zero weighted cycles using sp semiring



$$\mathbf{A} \quad = \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left[\begin{array}{ccc} \infty & 10 & 10 \\ \infty & \infty & 0 \\ \infty & 0 & \infty \end{array}\right] \end{array}$$

$\mathbf{A}^*$ $(= \mathbf{A} \oplus \mathbf{I}$ in this case) solves

$$\mathbf{X} = \mathbf{X}\mathbf{A} \oplus \mathbf{I}.$$

But so does this (dishonest) matrix!

$$\mathbf{F} \quad = \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left[\begin{array}{ccc} 0 & 9 & 9 \\ \infty & 0 & 0 \\ \infty & 0 & 0 \end{array}\right] \end{array}$$

For example :

$$\begin{aligned} & (\mathbf{FA})(0,1) \\ = \; & \min_{q \in \{0,1,2\}} \mathbf{F}(0,q) + \mathbf{A}(q,1) \\ = \; & \min(0+10, 9+\infty, 9+0) \\ = \; & 9 \\ = \; & \mathbf{F}(0,1) \end{aligned}$$

Goal: we need simple operations for constructing complex semirings

Example: elementary paths?

$G = (V,\ E)$. A semiring S, such that if A is an adjaceny matrix over S with

$$A(i,j) = \begin{cases} \{(i,j)\} & \text{if } (i,j) \in E \\ \{\} & \text{otherwise} \end{cases}$$

then

$$A^*(i,j) = \text{the set of all elementary (no loops) paths from } i \text{ to } j.$$

We could attempt to directly define such an algebra. But instead we will build it step-by-step using simple constructions ...

Lifted Product

Lifted product semigroup

Assume $(S, \otimes)$ is a semigroup. Let $\mathrm{lift}_{\times}(S) \equiv (\mathcal{P}_{\mathrm{fin}}(S), \hat{\otimes})$ where

$$X \hat{\otimes} Y = \{x \otimes y \mid x \in X,\ y \in Y\}$$

, where $X, Y \in \mathcal{P}_{\mathrm{fin}}(S)$, the set of finite subsets of S.

Lifted semiring

If $\overline{1}$ is the identity for $\otimes$, then

$$\mathrm{lift}(S) = (\mathcal{P}_{\mathrm{fin}}(S),\ \cup,\ \hat{\otimes},\ \{\},\ \{\overline{1}\})$$

is a semiring. Note that $\{\}$ is an annihilator for $\hat{\otimes}$.

When does $\mathrm{lift}(S)$ have an annihilator for $\cup$?

Reductions

If $(S, \oplus, \otimes)$ is a semiring and r is a function from S to S, then r is a reduction if for all a and b in S

1. $r(a) = r(r(a))$
2. $r(a \oplus b) = r(r(a) \oplus b) = r(a \oplus r(b))$
3. $r(a \otimes b) = r(r(a) \otimes b) = r(a \otimes r(b))$

Note that if either operation has an identity, then the first axioms is not needed. For example,

$$r(a) = r(a \oplus \overline{0}) = r(r(a) \oplus \overline{0}) = r(r(a))$$

Reduce operation

If $(S, \oplus, \otimes)$ is semiring and r is a reduction, then let $\mathrm{red}_r(S) = (S_r, \oplus_r, \otimes_r)$ where

1. $S_r = \{s \in S \mid r(s) = s\}$
2. $x \oplus_r y = r(x \oplus y)$
3. $x \otimes_r y = r(x \otimes y)$

Is the result always semiring?

Finally : A semiring of elementary paths

Semigroup of Sequences $\mathrm{seq}(X)$

- carrier : finite sequences over elements of X
- operation : concatenation
- identity : the empty string ϵ

Let X be a set of sequences over $\mathrm{lift}(\mathrm{seq}(E))$, and let

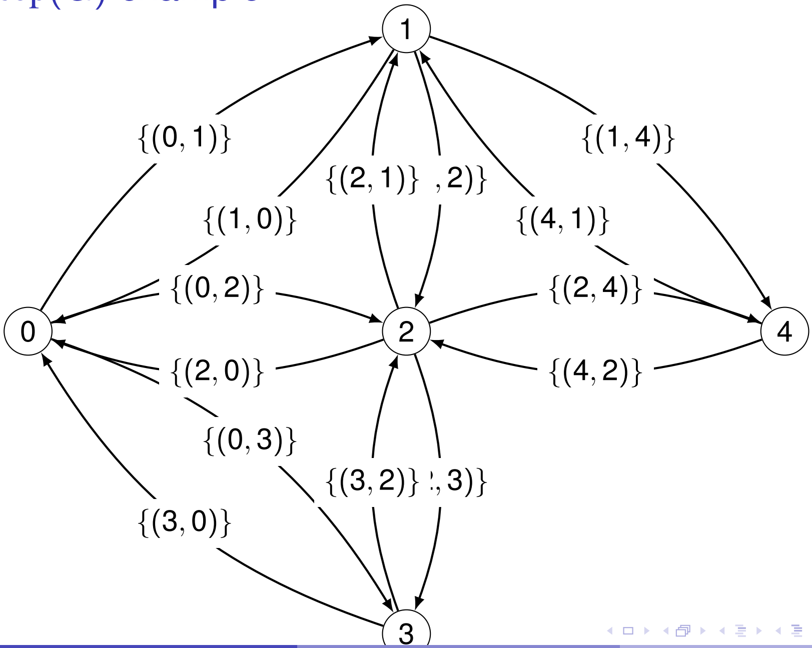
$$r(X) = \{p \in X \mid p \text{ is an elementary path in } G\}$$

Semiring of Elementary Paths

$$\mathrm{sep}(G) = \mathrm{red}_r(\mathrm{lift}(\mathrm{seq}(E)))$$

In order to check that $\mathrm{sep}(G)$ is indeed a semiring, we only need <u>understand</u> the functions $\mathrm{lift}(\_)$ and $\mathrm{red}\_(\_)$.

sep(G) example



sep(G) example, adjacency matrix

$$\mathbf{I} \quad = \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \{\epsilon\} & \{\} & \{\} & \{\} & \{\} \\ \{\} & \{\epsilon\} & \{\} & \{\} & \{\} \\ \{\} & \{\} & \{\epsilon\} & \{\} & \{\} \\ \{\} & \{\} & \{\} & \{\epsilon\} & \{\} \\ \{\} & \{\} & \{\} & \{\} & \{\epsilon\} \end{bmatrix} \end{array}$$

$$\mathbf{A} \quad = \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \{\} & \{[(0,1)]\} & \{[(0,2)]\} & \{[(0,3)]\} & \{\} \\ \{[(1,0)]\} & \{\} & \{[(1,2)]\} & \{\} & \{[(1,4)]\} \\ \{[(2,0)]\} & \{[(2,1)]\} & \{\} & \{[(2,3)]\} & \{[(2,4)]\} \\ \{[(3,0)]\} & \{\} & \{[(3,2)]\} & \{\} & \{\} \\ \{\} & \{[(4,1)]\} & \{[(4,2)]\} & \{\} & \{\} \end{bmatrix} \end{array}$$

Here I write a non-empty path p as $[p]$.

$\mathrm{sep}(G)$ example, solution

$$
\begin{array}{rcl}
\mathbf{A}^*(0,0) & = & \{\epsilon\} \\
\mathbf{A}^*(0,4) & = & \left\{
\begin{array}{l}
[(0,1),(1,4)], \\
[(0,1),(1,2),(2,4)], \\
[(0,2),(2,4)], \\
[(0,2),(2,1),(1,4)], \\
[(0,3),(3,2),(2,4)], \\
[(0,3),(3,2),(2,1),(1,4)]
\end{array}
\right\}
\end{array}
$$

More constructions: Direct Product of Semigroups

Let $(S, \oplus_S)$ and $(T, \oplus_T)$ be semigroups.

Definition (Direct product semigroup)

The direct product is denoted $(S, \oplus_S) \times (T, \oplus_T) = (S \times T, \oplus)$, where $\oplus = \oplus_S \times \oplus_T$ is defined as

$$(s_1, t_1) \oplus (s_2, t_2) = (s_1 \oplus_S s_2,\ t_1 \oplus_T t_2).$$

Lexicographic Product of Semigroups

Definition (Lexicographic product semigroup)

Suppose that semigroup $(S, \oplus_S)$ is commutative, idempotent, and selective and that $(T, \oplus_T)$ is a semigroup. The lexicographic product is denoted $(S, \oplus_S) \vec{\times} (T, \oplus_T) = (S \times T, \vec{\oplus})$, where $\vec{\oplus} = \oplus_S \vec{\times} \oplus_T$ is defined as

$$(s_1, t_1) \vec{\oplus} (s_2, t_2) = \begin{cases} (s_1 \oplus_S s_2, t_1 \oplus_T t_2) & s_1 = s_1 \oplus_S s_2 = s_2 \\ (s_1 \oplus_S s_2, t_1) & s_1 = s_1 \oplus_S s_2 \neq s_2 \\ (s_1 \oplus_S s_2, t_2) & s_1 \neq s_1 \oplus_S s_2 = s_2 \end{cases}$$

Lexicographic product of Bi-semigroups

$$(S,\ \oplus_S,\ \otimes_S)\ \vec{\times}\ (T,\ \oplus_T,\ \otimes_T) = (S \times T,\ \oplus_S\ \vec{\times}\ \oplus_T,\ \otimes_S \times \otimes_T)$$

Theorem

If $\oplus_S$ is commutative, idempotent, and selective, then

$$\mathrm{LD}(S\ \vec{\times}\ T) \iff \mathrm{LD}(S) \wedge \mathrm{LD}(T) \wedge (\mathrm{LC}(S) \vee \mathrm{LK}(T))$$

Where

Property	Definition
LD	$\forall a, b, c : c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$
LC	$\forall a, b, c : c \otimes a = c \otimes b \implies a = b$
LK	$\forall a, b, c : c \otimes a = c \otimes b$

Prove

$\text{LD}(S) \wedge \text{LD}(T) \wedge (\text{LC}(S) \vee \text{LK}(T)) \implies \text{LD}(S \vec{\times} T)$

Assume S and T are bisemigroups, $\text{LD}(S) \wedge \text{LD}(T) \wedge (\text{LC}(S) \vee \text{LK}(T))$, and

$$(s_1, t_1), (s_2, t_2), (s_3, t_3) \in S \times T.$$

Then (dropping operator subscripts for clarity) we have

$$\begin{aligned}
\text{lhs} &= (s_1, t_1) \otimes ((s_2, t_2) \vec{\oplus} (s_3, t_3)) \\
&= (s_1, t_1) \otimes (s_2 \oplus s_3, t_{\text{lhs}}) \\
&= (s_1 \otimes (s_2 \oplus s_3), t_1 \otimes t_{\text{lhs}})
\end{aligned}$$

$$\begin{aligned}
\text{rhs} &= ((s_1, t_1) \otimes (s_2, t_2)) \vec{\oplus} ((s_1, t_1) \otimes (s_3, t_3)) \\
&= (s_1 \otimes s_2, t_1 \otimes t_2) \vec{\oplus} (s_1 \otimes s_3, t_1 \otimes t_3) \\
&= ((s_1 \otimes s_2) \oplus_S (s_1 \otimes s_3), t_{\text{rhs}}) \\
&= (s_1 \otimes (s_2 \oplus s_3), t_{\text{rhs}})
\end{aligned}$$

where t_{lhs} and t_{rhs} are determined by the definition of $\vec{\oplus}$.

We need to show that $lhs = rhs$, that is $t_{\text{rhs}} = t_1 \otimes t_{\text{lhs}}$.

Case 1 : $\text{LC}(S)$

Note that we have

$$(\star) \quad \forall a, b, c : a \neq b \implies c \otimes a \neq c \otimes b$$

Case 1.1 : $s_2 = s_2 \oplus s_3 = s_3$. Then $t_{\text{lhs}} = t_2 \oplus t_3$ and $t_1 \otimes t_{\text{lhs}} = t_1 \otimes (t_2 \oplus t_3) = (t_1 \otimes t_2) \oplus (t_1 \otimes t_3)$, by $\text{LD}(S)$. Also, $s_1 \otimes_S s_2 = s_1 \otimes_S s_3$ and $s_1 \otimes s_2 = s_1 \otimes (s_2 \oplus s_3) = (s_1 \otimes s_2) \oplus (s_1 \otimes s_3)$, again by $\text{LD}(S)$. Therefore $t_{\text{rhs}} = (t_1 \otimes t_2) \oplus (t_1 \otimes t_3) = t_1 \otimes t_{\text{lhs}}$.

Case 1.2 : $s_2 = s_2 \oplus s_3 \neq s_3$. Then $t_1 \otimes t_{\text{lhs}} = t_1 \otimes t_2$ Also $s_2 = s_2 \oplus s_3 \implies s_1 \otimes s_2 = s_1 \otimes (s_2 \oplus s_3)$ and by $\star$ $s_2 \oplus s_3 \neq s_3 \implies s_1 \otimes (s_2 \oplus s_3) \neq s_1 \otimes s_3$. Thus, by $\text{LD}(S)$, $(s_1 \otimes s_2) \oplus (s_1 \otimes s_3) \neq s_1 \otimes s_3$ and we get $t_{\text{rhs}} = t_1 \otimes t_2 = t_1 \otimes t_{\text{lhs}}$.

Case 1.3 : $s_2 \neq s_2 \oplus_S s_3 = s_3$. Similar to case 1.2.

Case 2 : $\text{LK}(T)$

Case 2.1 : $s_2 = s_2 \oplus_S s_3 = s_3$. Same as Case 1.1.

Case 2.2 : $s_2 = s_2 \oplus_S s_3 \neq s_3$. Then $t_1 \otimes t_{\text{lhs}} = t_1 \otimes t_2$. Now, $(s_1 \otimes s_2) \oplus_S (s_1 \otimes s_3) = s_1 \otimes (s_2 \oplus s_3) = s_1 \otimes s_2$. So $t_{\text{rhs}} = (t_1 \otimes t_2) \oplus (t_1 \otimes t_3) = t_1 \otimes (t_2 \oplus t_3)$ or $t_{\text{rhs}} = (t_1 \otimes t_2)$. In either case, t_{rhs} is of the form $t_1 \otimes t$, so by $\text{LK}(T)$ we know that $t_{\text{rhs}} = t_1 \otimes t_{\text{lhs}}$.

Case 2.3 : $s_2 \neq s_2 \oplus_S s_3 = s_3$. Similar to case 2.2.

Examples

name	LD	LC	LK
min_plus	Yes	Yes	No
max_min	Yes	No	No
sep(G)	Yes	No	No

So we have

$$\mathsf{LD}(\text{min\_plus} \vec{\times} \text{max\_min})$$
$$\mathsf{LD}(\text{min\_plus} \vec{\times} \text{sep}(G))$$

But

$$\neg(\mathsf{LD}(\text{max\_min} \vec{\times} \text{min\_plus}))$$
$$\neg(\mathsf{LD}(\text{sep}(G) \vec{\times} \text{min\_plus}))$$

Operation for inserting a zero

$$\text{add\_zero}(\overline{0},\ (S,\ \oplus,\ \otimes)) = (S \uplus \{\overline{0}\}, \oplus_{\overline{0}},\ \otimes_{\overline{0}})$$

where

$$a \oplus_{\overline{0}} b \ = \ \begin{cases} a & (\text{if } b = \text{inr}(\overline{0})) \\ b & (\text{if } a = \text{inr}(\overline{0})) \\ \text{inl}(x \oplus y) & (\text{if } a = \text{inl}(x),\ b = \text{inl}(y)) \end{cases}$$

$$a \otimes_{\overline{0}} b \ = \ \begin{cases} \text{inr}(\overline{0}) & (\text{if } b = \text{inr}(\overline{0})) \\ \text{inr}(\overline{0}) & (\text{if } a = \text{inr}(\overline{0})) \\ \text{inl}(x \otimes y) & (\text{if } a = \text{inl}(x),\ b = \text{inl}(y)) \end{cases}$$

disjoint union

$$A \uplus B \equiv \{\text{inl}(a) \mid a \in A\} \cup \{\text{inr}(b) \mid b \in B\}$$

Operation for inserting a one

$$\text{add\_one}(\overline{1},\ (S,\ \oplus,\ \otimes)) = (S \uplus \{\overline{1}\}, \oplus_{\overline{1}},\ \otimes_{\overline{1}})$$

where

$$a \oplus_{\overline{1}} b \;=\; \begin{cases} \text{inr}(\overline{1}) & (\text{if } b = \text{inr}(\overline{1})) \\ \text{inr}(\overline{1}) & (\text{if } a = \text{inr}(\overline{1})) \\ \text{inl}(x \oplus y) & (\text{if } a = \text{inl}(x),\ b = \text{inl}(y)) \end{cases}$$

$$a \otimes_{\overline{1}} b \;=\; \begin{cases} a & (\text{if } b = \text{inr}(\overline{1})) \\ b & (\text{if } a = \text{inr}(\overline{1})) \\ \text{inl}(x \otimes y) & (\text{if } a = \text{inl}(x),\ b = \text{inl}(x)) \end{cases}$$

Shortest paths with best paths

Let's use

$$\text{add\_zero}(\infty,\ \text{min\_plus} \ \vec{\times}\ \text{sep}(G))$$

$$\mathbf{I} \;=\; \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \left[\begin{array}{ccccc} (0, \{\epsilon\}) & \infty & \infty & \infty & \infty \\ \infty & (0, \{\epsilon\}) & \infty & \infty & \infty \\ \infty & \infty & (0, \{\epsilon\}) & \infty & \infty \\ \infty & \infty & \infty & (0, \{\epsilon\}) & \infty \\ \infty & \infty & \infty & \infty & (0, \{\epsilon\}) \end{array}\right] \end{array}$$

$$\mathbf{A} \;=\; \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & \\ \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \left[\begin{array}{ccccc} \infty & (2, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (6, \{[(0,3)]\}) & \\ (2, \{[(1,0)]\}) & \infty & (5, \{[(1,2)]\}) & \infty & (4, \{[\\ (1, \{[(2,0)]\}) & (5, \{[(2,1)]\}) & \infty & (4, \{[(2,3)]\}) & (3, \{[\\ (6, \{[(3,0)]\}) & \infty & (4, \{[(3,2)]\}) & \infty & \\ \infty & (4, \{[(4,1)]\}) & (3, \{[(4,2)]\}) & \infty & \end{array}\right. \end{array}$$

Solution

$$\mathbf{A}^* = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad \end{array} \\ \begin{bmatrix} (0, \{\epsilon\}) & (2, \{[(0,1)]\}) & (1, \{[(0,2)]\}) \\ (2, \{[(1,0)]\}) & (0, \{\epsilon\}) & (3, \{[(1,0),(0,2)]\}) \\ (1, \{[(2,0)]\}) & (3, \{[(2,0),(0,1)]\}) & (0, \{\epsilon\}) \\ (5, \{[(3,2),(2,0)]\}) & (7, \{[(3,2),(2,0),(0,1)]\}) & (4, \{[(3,2)]\}) \\ (4, \{[(4,2),(2,0)]\}) & (4, \{[(4,1)]\}) & (3, \{[(4,2)]\}) \end{bmatrix} \end{array}$$