L11: Algebraic Path Problems with applications to Internet Routing
Lectures 02, 03

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Semigroups

Definition (Semigroup)

A semigroup $(S, \oplus)$ is a non-empty set S with a binary operation such that

$$\text{ASSOCIATIVE} \quad : \quad a \oplus (b \oplus c) = (a \oplus b) \oplus c$$

S	$\oplus$	where
$\mathbb{N}^{\infty}$	min	
$\mathbb{N}^{\infty}$	max	
$\mathbb{N}^{\infty}$	$+$	
2^W	$\cup$	
2^W	$\cap$	
S^*	$\circ$	$(abc \circ de = abcde)$
S	left	$(a \text{ left } b = a)$
S	right	$(a \text{ right } b = b)$

Special Elements

Definition

- $\alpha \in S$ is an identity if for all $a \in S$

$$a = \alpha \oplus a = a \oplus \alpha$$

- A semigroup is a monoid if it has an identity.
- ω is an annihilator if for all $a \in S$

$$\omega = \omega \oplus a = a \oplus \omega$$

S	$\oplus$	α	ω
$\mathbb{N}^{\infty}$	min	∞	0
$\mathbb{N}^{\infty}$	max	0	∞
$\mathbb{N}^{\infty}$	$+$	0	∞
2^W	$\cup$	$\{\}$	W
2^W	$\cap$	W	$\{\}$
S^*	$\circ$	ϵ	
S	left		
S	right		

Important Properties

Definition (Some Important Semigroup Properties)

$$\begin{array}{rcrcl} \text{COMMUTATIVE} & : & a \oplus b & = & b \oplus a \\ \text{SELECTIVE} & : & a \oplus b & \in & \{a,\ b\} \\ \text{IDEMPOTENT} & : & a \oplus a & = & a \end{array}$$

S	$\oplus$	COMMUTATIVE	SELECTIVE	IDEMPOTENT
$\mathbb{N}^\infty$	min	$\star$	$\star$	$\star$
$\mathbb{N}^\infty$	max	$\star$	$\star$	$\star$
$\mathbb{N}^\infty$	$+$	$\star$		
2^W	$\cup$	$\star$		$\star$
2^W	$\cap$	$\star$		$\star$
S^*	$\circ$			
S	left		$\star$	$\star$
S	right		$\star$	$\star$

Order Relations

We are interested in order relations $\leq \subseteq S \times S$

Definition (Important Order Properties)

REFLEXIVE	:	$a \leq a$
TRANSITIVE	:	$a \leq b \wedge b \leq c \rightarrow a \leq c$
ANTISYMMETRIC	:	$a \leq b \wedge b \leq a \rightarrow a = b$
TOTAL	:	$a \leq b \vee b \leq a$

	pre-order	partial order	preference order	total order
REFLEXIVE	$\star$	$\star$	$\star$	$\star$
TRANSITIVE	$\star$	$\star$	$\star$	$\star$
ANTISYMMETRIC		$\star$		$\star$
TOTAL			$\star$	$\star$

Canonical Pre-order of a Commutative Semigroup

Suppose $\oplus$ is commutative.

Definition (Canonical pre-orders)

$$
\begin{array}{lcl}
a \trianglelefteq_{\oplus}^{R} b & \equiv & \exists c \in S : b = a \oplus c \\
a \trianglelefteq_{\oplus}^{L} b & \equiv & \exists c \in S : a = b \oplus c
\end{array}
$$

Lemma (Sanity check)

Associativity of $\oplus$ implies that these relations are transitive.

Proof.

Note that $a \trianglelefteq_{\oplus}^{R} b$ means $\exists c_1 \in S : b = a \oplus c_1$, and $b \trianglelefteq_{\oplus}^{R} c$ means $\exists c_2 \in S : c = b \oplus c_2$. Letting $c_3 = c_1 \oplus c_2$ we have $c = b \oplus c_2 = (a \oplus c_1) \oplus c_2 = a \oplus (c_1 \oplus c_2) = a \oplus c_3$. That is, $\exists c_3 \in S : c = a \oplus c_3$, so $a \trianglelefteq_{\oplus}^{R} c$. The proof for $\trianglelefteq_{\oplus}^{L}$ is similar. $\square$

Canonically Ordered Semigroup

Definition (Canonically Ordered Semigroup)

A commutative semigroup $(S, \oplus)$ is canonically ordered when $a \trianglelefteq_{\oplus}^{R} c$ and $a \trianglelefteq_{\oplus}^{L} c$ are partial orders.

Definition (Groups)

A monoid is a group if for every $a \in S$ there exists a $a^{-1} \in S$ such that $a \oplus a^{-1} = a^{-1} \oplus a = \alpha$.

Canonically Ordered Semigroups vs. Groups

Lemma (THE BIG DIVIDE)

Only a trivial group is canonically ordered.

Proof.

If $a,\ b \in S$, then $a = \alpha_{\oplus} \oplus a = (b \oplus b^{-1}) \oplus a = b \oplus (b^{-1} \oplus a) = b \oplus c$, for $c = b^{-1} \oplus a$, so $a \trianglelefteq^{L}_{\oplus} b$. In a similar way, $b \trianglelefteq^{R}_{\oplus} a$. Therefore $a = b$. $\square$

Natural Orders

Definition (Natural orders)

Let $(S, \oplus)$ be a semigroup.

$$
\begin{array}{lcl}
a \leq^{L}_{\oplus} b & \equiv & a = a \oplus b \\
a \leq^{R}_{\oplus} b & \equiv & b = a \oplus b
\end{array}
$$

Lemma

If $\oplus$ is commutative and idempotent, then $a \trianglelefteq^{D}_{\oplus} b \iff a \leq^{D}_{\oplus} b$, for $D \in \{R, L\}$.

Proof.

$$
\begin{array}{lcl}
a \trianglelefteq^{R}_{\oplus} b & \iff & b = a \oplus c = (a \oplus a) \oplus c = a \oplus (a \oplus c) \\
& = & a \oplus b \iff a \leq^{R}_{\oplus} b \\
a \trianglelefteq^{L}_{\oplus} b & \iff & a = b \oplus c = (b \oplus b) \oplus c = b \oplus (b \oplus c) \\
& = & b \oplus a = a \oplus b \iff a \leq^{L}_{\oplus} b
\end{array}
$$

$\square$

Special elements and natural orders

Lemma (Natural Bounds)

- *If α exists, then for all a, $a \leq_{\oplus}^{L} \alpha$ and $\alpha \leq_{\oplus}^{R} a$*
- *If ω exists, then for all a, $\omega \leq_{\oplus}^{L} a$ and $a \leq_{\oplus}^{R} \omega$*
- *If α and ω exist, then S is bounded.*

$$\begin{array}{ccccc} \omega & \leq_{\oplus}^{L} & a & \leq_{\oplus}^{L} & \alpha \\ \alpha & \leq_{\oplus}^{R} & a & \leq_{\oplus}^{R} & \omega \end{array}$$

Remark (Thanks to Iljitsch van Beijnum)

Note that this means for (min, $+$) we have

$$\begin{array}{ccccc} 0 & \leq_{\min}^{L} & a & \leq_{\min}^{L} & \infty \\ \infty & \leq_{\min}^{R} & a & \leq_{\min}^{R} & 0 \end{array}$$

and still say that this is bounded, even though one might argue with the terminology!

Examples of special elements

S	$\oplus$	α	ω	$\leq^{\mathrm{L}}_{\oplus}$	$\leq^{\mathrm{R}}_{\oplus}$
$\mathbb{N} \cup \{\infty\}$	min	∞	0	$\leq$	$\geq$
$\mathbb{N} \cup \{\infty\}$	max	0	∞	$\geq$	$\leq$
$\mathcal{P}(W)$	$\cup$	$\{\}$	W	$\subseteq$	$\supseteq$
$\mathcal{P}(W)$	$\cap$	W	$\{\}$	$\supseteq$	$\subseteq$

Property Management

Lemma

Let $D \in \{R,\ L\}$.

1. $\text{IDEMPOTENT}((S,\ \oplus)) \iff \text{REFLEXIVE}((S,\ \leq^{D}_{\oplus}))$
2. $\text{COMMUTATIVE}((S,\ \oplus)) \implies \text{ANTISYMMETRIC}((S,\ \leq^{D}_{\oplus}))$
3. $\text{COMMUTATIVE}((S,\ \oplus)) \implies (\text{SELECTIVE}((S,\ \oplus)) \iff \text{TOTAL}((S,\ \leq^{D}_{\oplus})))$

Proof.

1. $a \leq^{D}_{\oplus} a \iff a = a \oplus a,$
2. $a \leq^{L}_{\oplus} b \wedge b \leq^{L}_{\oplus} a \iff a = a \oplus b \wedge b = b \oplus a \implies a = b$
3. $a = a \oplus b \vee b = a \oplus b \iff a \leq^{L}_{\oplus} b \vee b \leq^{L}_{\oplus} a$

$\square$

Bounds

Suppose $(S, \leq)$ is a partially ordered set.

greatest lower bound

For $a, b \in S$, the element $c \in S$ is the greatest lower bound of a and b, written $c = a \operatorname{glb} b$, if it is a lower bound ($c \leq a$ and $c \leq b$), and for every $d \in S$ with $d \leq a$ and $d \leq b$, we have $d \leq c$.

least upper bound

For $a, b \in S$, the element $c \in S$ is the least upper bound of a and b, written $c = a \operatorname{lub} b$, if it is an upper bound ($a \leq c$ and $b \leq c$), and for every $d \in S$ with $a \leq d$ and $b \leq d$, we have $c \leq d$.

Semi-lattices

Suppose $(S, \leq)$ is a partially ordered set.

meet-semilattice

S is a meet-semilattice if $a \; \mathrm{glb} \; b$ exists for each $a, \; b \in S$.

join-semilattice

S is a join-semilattice if $a \; \mathrm{lub} \; b$ exists for each $a, \; b \in S$.

Fun Facts

Fact 1

Suppose $(S, \oplus)$ is a commutative and idempotent semigroup.

- $(S, \leq_{\oplus}^{L})$ is a meet-semilattice with $a \,\mathrm{glb}\, b = a \oplus b$.
- $(S, \leq_{\oplus}^{R})$ is a join-semilattice with $a \,\mathrm{lub}\, b = a \oplus b$.

Fact 2

Suppose $(S, \leq)$ is a partially ordered set.

- If $(S, \leq)$ is a meet-semilattice, then (S, glb) is a commutative and idempotent semigroup.
- If $(S, \leq)$ is a join-semilattice, then (S, lub) is a commutative and idempotent semigroup.

That is, semi-lattices represent the same class of structures as commutative and idempotent semigroups.

Bi-semigroups and Pre-Semirings

$(S, \oplus, \otimes)$ is a bi-semigroup when

- $(S, \oplus)$ is a semigroup
- $(S, \otimes)$ is a semigroup

$(S, \oplus, \otimes)$ is a pre-semiring when

- $(S, \oplus, \otimes)$ is a bi-semigroup
- $\oplus$ is commutative

and left- and right-distributivity hold,

$$\begin{array}{lcrcl} \mathrm{LD} & : & a \otimes (b \oplus c) & = & (a \otimes b) \oplus (a \otimes c) \\ \mathrm{RD} & : & (a \oplus b) \otimes c & = & (a \otimes c) \oplus (b \otimes c) \end{array}$$

Semirings

$(S, \oplus, \otimes, \overline{0}, \overline{1})$ is a semiring when

- $(S, \oplus, \otimes)$ is a pre-semiring
- $(S, \oplus, \overline{0})$ is a (commutative) monoid
- $(S, \otimes, \overline{1})$ is a monoid
- $\overline{0}$ is an annihilator for $\otimes$

Examples

Pre-semirings

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
min_plus	$\mathbb{N}$	min	$+$		0
max_min	$\mathbb{N}$	max	min	0	

Semirings

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
sp	$\mathbb{N}^{\infty}$	min	$+$	∞	0
bw	$\mathbb{N}^{\infty}$	max	min	0	∞

Note the sloppiness — the symbols $+$, max, and min in the two tables represent different functions....

How about $(\max, +)$?

Pre-semiring

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
$\mathrm{max\_plus}$	$\mathbb{N}$	max	$+$	0	0

- What about "$\overline{0}$ is an annihilator for $\otimes$"? No!

Semiring $(\mathrm{max\_plus}^{-\infty} = \mathrm{add\_zero}(-\infty, \ \mathrm{max\_min}))$

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
$\mathrm{max\_plus}^{-\infty}$	$\mathbb{N} \cup \{-\infty\}$	max	$+$	$-\infty$	0

Matrix Semirings

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ a semiring
- Define the semiring of $n \times n$-matrices over S : $(\mathbb{M}_n(S), \oplus, \otimes, \mathbf{J}, \mathbf{I})$

$\oplus$ and $\otimes$

$$
\begin{aligned}
(\mathbf{A} \oplus \mathbf{B})(i, j) &= \mathbf{A}(i, j) \oplus \mathbf{B}(i, j) \\
(\mathbf{A} \otimes \mathbf{B})(i, j) &= \bigoplus_{1 \le q \le n} \mathbf{A}(i, q) \otimes \mathbf{B}(q, j)
\end{aligned}
$$

J and I

$$
\begin{aligned}
\mathbf{J}(i, j) &= \overline{0} \\
\mathbf{I}(i, j) &= \begin{cases} \overline{1} & (\text{if } i = j) \\ \overline{0} & (\text{otherwise}) \end{cases}
\end{aligned}
$$

$\mathbb{M}_n(S)$ is a semiring!

For example, here is left distribution

$$\mathbf{A} \otimes (\mathbf{B} \oplus \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \oplus (\mathbf{A} \otimes \mathbf{C})$$

$$\begin{aligned}
& (\mathbf{A} \otimes (\mathbf{B} \oplus \mathbf{C}))(i,\ j) \\
= & \bigoplus_{1 \leq q \leq n} \mathbf{A}(i,\ q) \otimes (\mathbf{B} \oplus \mathbf{C})(q,\ j) \\
= & \bigoplus_{1 \leq q \leq n} \mathbf{A}(i,\ q) \otimes (\mathbf{B}(q,\ j) \oplus \mathbf{C}(q,\ j)) \\
= & \bigoplus_{1 \leq q \leq n} (\mathbf{A}(i,\ q) \otimes \mathbf{B}(q,\ j)) \oplus (\mathbf{A}(i,\ q) \otimes \mathbf{C}(q,\ j)) \\
= & (\bigoplus_{1 \leq q \leq n} \mathbf{A}(i,\ q) \otimes \mathbf{B}(q,\ j)) \oplus (\bigoplus_{1 \leq q \leq n} \mathbf{A}(i,\ q) \otimes \mathbf{C}(q,\ j)) \\
= & ((\mathbf{A} \otimes \mathbf{B}) \oplus (\mathbf{A} \otimes \mathbf{C}))(i,\ j)
\end{aligned}$$

Note : we only needed left-distributivity on S.

Matrix encoding path problems

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ a semiring
- $G = (V, E)$ a directed graph
- $w \in E \to S$ a weight function

Path weight

The weight of a path $p = i_1, i_2, i_3, \cdots, i_k$ is

$$w(p) = w(i_1, i_2) \otimes w(i_2, i_3) \otimes \cdots \otimes w(i_{k-1}, i_k).$$

The empty path is given the weight $\overline{1}$.

Adjacency matrix **A**

$$\mathbf{A}(i, j) = \begin{cases} w(i, j) & \text{if } (i, j) \in E, \\ \overline{0} & \text{otherwise} \end{cases}$$

The general problem of finding globally optimal paths

Given an adjacency matrix $\mathbf{A}$, find $\mathbf{R}$ such that for all $i, j \in V$

$$\mathbf{R}(i, j) = \bigoplus_{p \in P(i, j)} w(p)$$

How can we solve this problem?

Matrix methods

Matrix powers, $\mathbf{A}^k$

$$\begin{aligned}\mathbf{A}^0 &= \mathbf{I} \\ \mathbf{A}^{k+1} &= \mathbf{A} \otimes \mathbf{A}^k\end{aligned}$$

Closure, $\mathbf{A}^*$

$$\begin{aligned}\mathbf{A}^{(k)} &= \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k \\ \mathbf{A}^* &= \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k \oplus \cdots\end{aligned}$$

Note: $\mathbf{A}^*$ might not exist. Why?

Matrix methods can compute optimal path weights

- Let $P(i, j)$ be the set of paths from i to j.
- Let $P^k(i, j)$ be the set of paths from i to j with exactly k arcs.
- Let $P^{(k)}(i, j)$ be the set of paths from i to j with at most k arcs.

Theorem

$$
\begin{array}{lrcl}
(1) & \mathbf{A}^k(i,\ j) & = & \bigoplus_{p \in P^k(i,\ j)} w(p) \\
(2) & \mathbf{A}^{(k)}(i,\ j) & = & \bigoplus_{p \in P^{(k)}(i,\ j)} w(p) \\
(3) & \mathbf{A}^*(i,\ j) & = & \bigoplus_{p \in P(i,\ j)} w(p)
\end{array}
$$

Warning again: for some semirings the expression $\mathbf{A}^*(i,\ j)$ might not be well-defeind. Why?

Proof of (1)

By induction on k. Base Case: $k = 0$.

$$P^0(i,\ i) = \{\epsilon\},$$

so $\mathbf{A}^0(i\ , i) = \mathbf{I}(i,\ i) = \overline{1} = w(\epsilon)$.

And $i \neq j$ implies $P^0(i, j) = \{\}$. By convention

$$\bigoplus_{p \in \{\}} w(p) = \overline{0} = \mathbf{I}(i,\ j).$$

Proof of (1)

Induction step.

$$
\begin{aligned}
\mathbf{A}^{k+1}(i,j) &= (\mathbf{A} \otimes \mathbf{A}^{k})(i,\ j) \\
&= \bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes \mathbf{A}^{k}(q,\ j) \\
&= \bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes \Big(\bigoplus_{p \in P^{k}(q,\ j)} w(p) \Big) \\
&= \bigoplus_{1 \le q \le n} \bigoplus_{p \in P^{k}(q,\ j)} \mathbf{A}(i,\ q) \otimes w(p) \\
&= \bigoplus_{(i,\ q) \in E} \bigoplus_{p \in P^{k}(q,j)} w(i,\ q) \otimes w(p) \\
&= \bigoplus_{p \in P^{k+1}(i,\ j)} w(p)
\end{aligned}
$$

When does $\mathbf{A}^{(*)}$ exist? Try a general approach.

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ a semiring

Powers, a^k

$$\begin{aligned} a^0 &= \overline{1} \\ a^{k+1} &= a \otimes a^k \end{aligned}$$

Closure, a^*

$$\begin{aligned} a^{(k)} &= a^0 \oplus a^1 \oplus a^2 \oplus \cdots \oplus a^k \\ a^* &= a^0 \oplus a^1 \oplus a^2 \oplus \cdots \oplus a^k \oplus \cdots \end{aligned}$$

Definition (q stability)

If there exists a q such that $a^{(q)} = a^{(q+1)}$, then a is q-stable. Therefore, $a^* = a^{(q)}$, assuming $\oplus$ is idempotent.

More Fun Facts

Fact 3

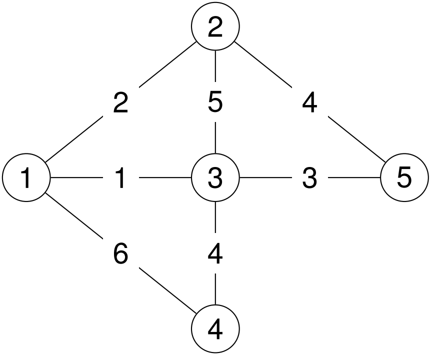
If $\overline{1}$ is an annihiltor for $\oplus$, then every $a \in S$ is 0-stable!

Fact 4

If S is 0-stable, then $\mathbb{M}_n(S)$ is $(n-1)$-stable. That is,

$$\mathbf{A}^* = \mathbf{A}^{(n-1)} = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^{n-1}$$

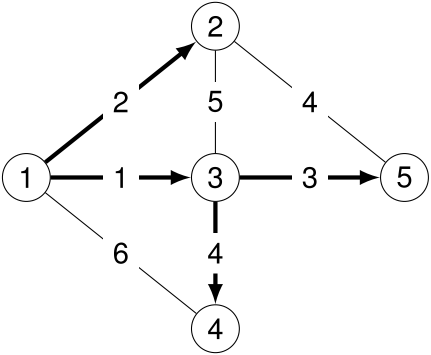
Shortest paths example, $(\mathbb{N}^{\infty}, \min, +)$



The adjacency matrix

$$
\mathbf{A} = \begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{ccccc}
\infty & 2 & 1 & 6 & \infty \\
2 & \infty & 5 & \infty & 4 \\
1 & 5 & \infty & 4 & 3 \\
6 & \infty & 4 & \infty & \infty \\
\infty & 4 & 3 & \infty & \infty
\end{array}\right]
\end{array}
$$

Shortest paths example, $(\mathbb{N}^{\infty},\ \min,\ +)$



Bold arrows indicate the shortest-path tree rooted at 1.

The routing matrix

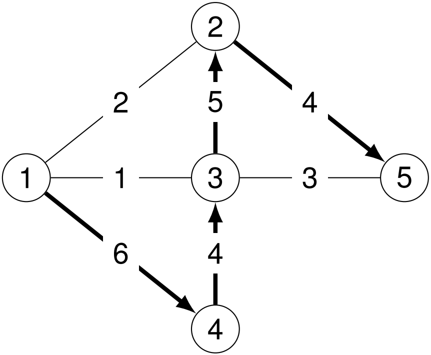
$$
\mathbf{R} \quad = \quad \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{bmatrix} \end{array}
$$

Matrix **R** solves this global optimality problem:

$$
\mathbf{R}(i,\ j) = \min_{p \in P(i,\ j)} w(p),
$$

where $P(i,\ j)$ is the set of all paths from i to j.

Widest paths example, $(\mathbb{N}^{\infty}, \max, \min)$



Bold arrows indicate the widest-path tree rooted at 1.

The routing matrix

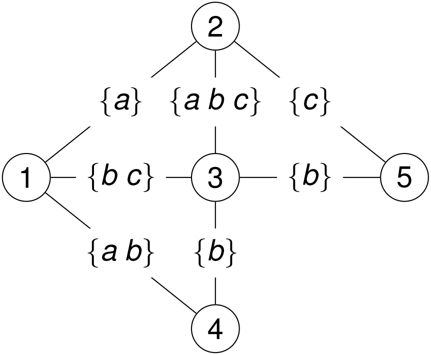
$$\mathbf{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \infty & 4 & 4 & 6 & 4 \\ 4 & \infty & 5 & 4 & 4 \\ 4 & 5 & \infty & 4 & 4 \\ 6 & 4 & 4 & \infty & 4 \\ 4 & 4 & 4 & 4 & \infty \end{bmatrix} \end{array}$$

Matrix **R** solves this global optimality problem:

$$\mathbf{R}(i, j) = \max_{p \in P(i, j)} w(p),$$

where $w(p)$ is now the minimal edge weight in p.

Unfamiliar example, $(2^{\{a, b, c\}}, \cup, \cap)$



We want a Matrix **R** to solve this global optimality problem:

$$\mathbf{R}(i, j) = \bigcup_{p \in P(i, j)} w(p),$$

where $w(p)$ is now the intersection of all edge weights in p.

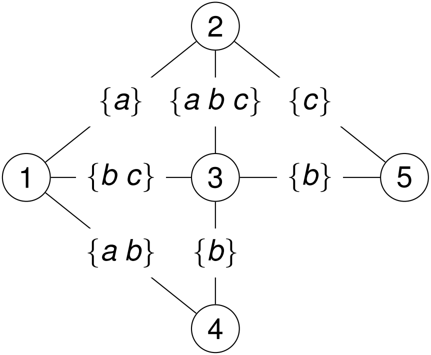
For $x \in \{a, b, c\}$, interpret $x \in \mathbf{R}(i, j)$ to mean that there is at least one path from i to j with x in every arc weight along the path.

Unfamiliar example, $(2^{\{a,\ b,\ c\}},\ \cup,\ \cap)$

The matrix **R**

$$
\begin{array}{c c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{bmatrix}
\{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\} & \{b\ c\} \\
\{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\} & \{b\ c\} \\
\{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\} & \{b\ c\} \\
\{a\ b\} & \{a\ b\} & \{a\ b\} & \{a\ b\ c\} & \{b\} \\
\{b\ c\} & \{b\ c\} & \{b\ c\} & \{b\} & \{a\ b\ c\}
\end{bmatrix}
\end{array}
$$

Another unfamiliar example, $(2^{\{a,\ b,\ c\}},\ \cap,\ \cup)$



We want matrix $\mathbf{R}$ to solve this global optimality problem:

$$\mathbf{R}(i,\ j) = \bigcap_{p \in P(i,\ j)} w(p),$$

where $w(p)$ is now the union of all edge weights in p.

For $x \in \{a,\ b,\ c\}$, interpret $x \in \mathbf{R}(i,\ j)$ to mean that every path from i to j has at least one arc with weight containing x.

Another unfamiliar example, $(2^{\{a,\ b,\ c\}},\ \cap,\ \cup)$

The matrix **R**

$$
\begin{array}{c}
\\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \;\;1\;\; & \;\;2\;\; & \;\;3\;\; & \;\;4\;\; & \;\;5\;\; \end{array} \\
\left[\begin{array}{ccccc}
\{\} & \{\} & \{b\} & \{b\} & \{\} \\
\{\} & \{\} & \{b\} & \{b\} & \{\} \\
\{b\} & \{b\} & \{\} & \{b\} & \{b\} \\
\{b\} & \{b\} & \{b\} & \{\} & \{b\} \\
\{\} & \{\} & \{b\} & \{b\} & \{\}
\end{array}\right]
\end{array}
$$

Homework number 1

- Prove that matrix multiplication (slide 20) is associative.
- Prove Fun Facts 1 and 2 (see slide 15)
- Prove Fun Facts 3 and 4 (see slide 29)