Inductively defined subsets

Given a set of axioms and rules over a set $U$, the subset of $U$ inductively defined by the axioms and rules consists of all and only the elements $u \in U$ for which there is a derivation with conclusion $u$.

(In fact $u \in \{a, b\}^*$ is in the subset if and only if it contains the same number of $a$ and $b$ symbols.)
Rule Induction

**Theorem.** The subset $I \subseteq U$ inductively defined by a collection of axioms and rules is closed under them and is the least such subset: if $S \subseteq U$ is also closed under the axioms and rules, then $I \subseteq S$.

Given axioms and rules for inductively defining a subset of a set $U$, we say that a subset $S \subseteq U$ is **closed under the axioms and rules** if

- for every axiom $a$, it is the case that $a \in S$,
- for every rule $h_1 h_2 \cdots h_n$, if $h_1, h_2, \ldots, h_n \in S$, then $c \in S$. 
E.g. for the axioms & rules:

\[ \frac{u}{\Sigma} \quad \frac{u}{u v a} \quad \frac{u v}{u v} \]

(all \( u, v \in \{a, b\}^* \))

The subset:

\[ \{ u \in \{a, b\}^* \mid \#_a(u) = \#_b(u) \} \]

number of 'a's in the string u
E.g. for the axiom & rules

<table>
<thead>
<tr>
<th></th>
<th>( u )</th>
<th>( u' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E )</td>
<td>( a, b )</td>
<td>( b, a )</td>
</tr>
</tbody>
</table>

\( \frac{u}{\text{uv}} \) (all \( u, v \in \{a, b\}^* \))

The subset

\[ \{ u \in \{a, b\}^* \mid \#_a(u) = \#_b(u) \} \]

is closed under the axiom & rules.

(\( \text{so is } \{ u \in \{a, b\}^* \mid \text{length of } u \text{ is even } \} \)
N.B. for a given set $R$ of axioms & rules

\[
\{ u \in U \mid \forall S \subseteq U. (S \text{ closed under } R) \Rightarrow u \in S \}
\]

is closed under $R$ (Why?)
and so is the smallest such (with respect to subset inclusion, $\subseteq$).
Theorem. The subset $I \subseteq U$ inductively defined by a collection of axioms and rules is \textit{closed} under them and is the least such subset: if $S \subseteq U$ is also closed under the axioms and rules, then $I \subseteq S$. 

"the least subset closed under the axioms & rules" is sometimes taken as the definition of "inductively defined subset"
Proof of the Theorem

- I is closed under each axiom $a$

'cos $a$ is a derivation witnessing $a \in I$

tree with root = leaf = $a$
Proof of the Theorem

\( I \) is closed under each rule

\[
\frac{h_1 \ldots h_n}{C}, \text{cos if } h_1 \in I \& \ldots \& h_n \in I
\]

then there are derivations

\[
\begin{align*}
D_1 & \quad \ldots \quad D_n \\
h_1 & \quad h_n
\end{align*}
\]
Proof of the Theorem

- $I$ is closed under each rule
  
  $h_1, ..., h_n \implies \cos$ if $h_1 \in I \land ... \land h_n \in I$

Then there are derivations

- is a derivation witnessing that $c \in I$
Proof of the Theorem

So $I$ is closed under the axioms & rules.

Finally, need to show for any $S \subseteq U$

$S$ closed under axioms & rules implies $I \subseteq S$
Proof of the Theorem

Suppose $S$ closed under axioms & rules. For each $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, let

$P(n) =$ "all derivations of height $n$ have their conclusion in $S$"

To see $I \subseteq S$, suffices to show

$$\forall n \in \mathbb{N}. \ P(n)$$

Prove this by Mathematical Induction
Proof of the Theorem

$P(n) = \"all derivations of height n have their conclusion in S\"$

$\forall n \in \mathbb{N}, P(n)$

- Base case $P(0)$ - trivial $\checkmark$
Proof of the Theorem

\[ P(n) = \text{"all derivations of height } n \text{ have their conclusion in } S" \]

\[ \forall n \in \mathbb{N}, P(n) \]

- Base case \( P(0) \) - trivial
- Induction step \( P(n) \Rightarrow P(n+1) \)
Proof of the Theorem

\[ P(n) = "\text{all derivations of height } n \text{ have their conclusion in } S" \]

- Induction step \( P(n) \Rightarrow P(n+1) \)

Suppose \( P(n) \) & \( D \) is a derivation of height \( n+1 \), with conclusion \( c \) say.

So \( D \) looks like

\[
\begin{array}{c}
\Delta_1 \\
\vdots \\
\Delta_m \\
\end{array}
\]

This is one of the rules (or an axiom, if \( m = 0 \))

\[
\frac{c_1 \ldots c_m}{C}
\]
Proof of the Theorem

\[ P(n) = "all derivations of height n have their conclusion in S" \]

- Induction step \( P(n) \Rightarrow P(n+1) \)

Suppose \( P(n) \) & \( D \) is a derivation of height \( n+1 \), with conclusion \( c \) say.

So \( D \) looks like:

\[ \begin{array}{c}
\cdots \\
D_1 \\
\vdots \\
\cdots \\
D_m \\
\end{array} \]

\[ \{ \begin{array}{c}
c_1 \\
\vdots \\
c_m \\
\end{array} \} \text{ height } n \]

This is one of the rules (or an axiom, if \( m = 0 \))
Proof of the Theorem

\[ P(n) = "\text{all derivations of height } n \text{ have their conclusion in } S" \]

- Induction step: \( P(n) \Rightarrow P(n+1) \)

Suppose \( P(n) \) & \( D \) is a derivation of height \( n+1 \), with conclusion \( c \) say.

So \( D \) looks like

\[
\begin{align*}
\ & \ c
\ & \ c_1 \ldots \ c_m \ \ & \ \in \ S \ \ & \ \in \ S \ \ & \ \text{height } n \\
\ & \ \underline{c}
\end{align*}
\]
Proof of the Theorem

\[ P(n) = \text{"all derivations of height } n \text{ have their conclusion in } S\]"

- Induction step \[ P(n) \implies P(n+1) \]

Suppose \( P(n) \) & \( D \) is a derivation of height \( n+1 \), with conclusion \( c \) say.

So \( D \) looks like

\[
\begin{array}{c}
    c \\
    \in S \\
    \in S
\end{array}
\]_\text{height}_n

\[
\begin{array}{c}
    c_1 \\
    \in S
\end{array}
\]_m

This is one of the rules under which \( S \) is closed,

so

\[
\begin{array}{c}
    c \\
    \in S
\end{array}
\]
Proof of the Theorem

\[ P(n) = \text{"all derivations of height } n \text{ have their conclusion in } S\]

- Induction step: \( P(n) \Rightarrow P(n+1) \)
  Suppose \( P(n) \) & \( D \) \n  \text{height } n+1, \text{ with conclusion } c

\[ \text{... we have proved } P(n+1). \quad \checkmark \]
Theorem. The subset $I \subseteq U$ inductively defined by a collection of axioms and rules is closed under them and is the least such subset: if $S \subseteq U$ is also closed under the axioms and rules, then $I \subseteq S$.

We use the theorem as method of proof: given a property $P(u)$ of elements of $U$, to prove $\forall u \in I. P(u)$ it suffices to show

- **base cases:** $P(a)$ holds for each axiom $a$

- **induction steps:** $P(h_1) \& P(h_2) \& \cdots \& P(h_n) \Rightarrow P(c)$ holds for each rule $h_1 h_2 \cdots h_n c$

(To see this, apply the theorem with $S = \{u \in U \mid P(u)\}$.)
Example using rule induction

Let $I$ be the subset of $\{a, b\}^*$ inductively defined by the axioms and rules on Slide 15.

Associated Rule Induction:

- $P(\varepsilon)$
- $\forall u \in I. \ P(u) \Rightarrow P(au) \Rightarrow P(uv)$
- $\forall u \in I. \ P(u) \Rightarrow P(bu) \Rightarrow P(uv)$
- $\forall u, v \in I. \ P(u) \& P(v) \Rightarrow P(uv)$

$\Rightarrow \forall u \in I. \ P(u)$
Example using rule induction

Let $I$ be the subset of $\{a, b\}^*$ inductively defined by the axioms and rules on Slide 15.

For $u \in \{a, b\}^*$, let $P(u)$ be the property

$u$ contains the same number of $a$ and $b$ symbols

We can prove $\forall u \in I. P(u)$ by rule induction:

- base case: $P(\varepsilon)$ is true (the number of $a$s and $b$s is zero!)
- induction steps: if $P(u)$ and $P(v)$ hold, then clearly so do $P(aub)$, $P(bua)$ and $P(uv)$.

(It’s not so easy to show $\forall u \in \{a, b\}^*. P(u) \Rightarrow u \in I$ – rule induction for $I$ is not much help for that.)
Example [CST 2009, Paper2, Question 5]

$I \subseteq \{a,b\}^*$ inductively defined by

\[
\begin{array}{cccc}
a & 0 & u & u \\hline
\vphantom{u}a & au & buv
\end{array}
\]
Example [CST 2009, Paper2, Question 5]

\[ I \subseteq \{a, b\}^* \text{ inductively defined by} \]

\[
\begin{array}{ccc}
  \\
  a & u & uv \\
  a u & & b u v \\
\end{array}
\]

In this case, Rule Induction says:

if (0) \( P(a) \)

& (1) \( \forall u \in I. \ P(u) \Rightarrow P(au) \)

& (2) \( \forall u, v \in I. \ P(u) \land P(v) \Rightarrow P(buv) \)

then \( \forall u \in I. \ P(u) \)
Example [CST 2009, Paper2, Question 5]

\[ I \subseteq \{a,b\}^* \text{ inductively defined by} \]

\[
\begin{array}{c c c}
  a & u & u \\
  \_ & au & buv
\end{array}
\]

Want to show

\[ u \in I \Rightarrow \#_a(u) > \#_b(u) \]

number of 'a's in the string u
Example [CST 2009, Paper2, Question 5]

$I \subseteq \{a,b\}^*$ inductively defined by

\[
\begin{array}{c c c c}
\text{a} & 0 & \text{u} & \text{u} \\
\text{au} & \text{bu} & \text{uv} & 2
\end{array}
\]

Want to show

\[ u \in I \Rightarrow \#_a(u) > \#_b(u) \]

Do so by Rule Induction, with property

\[ P(u) = \#_a(u) > \#_b(u) \]
Example [CST 2009, Paper 2, Question 5]

$I \subseteq \{a, b\}^*$ inductively defined by

\[
\begin{array}{c|cc|c}
\text{a} & \text{u} & \text{uv} \\
\text{au} & \text{u} & \text{uv} \\
\end{array}
\]

\[P(u) = \#_a(u) > \#_b(u)\]

(i) \(P(a)\) holds (1 > 0)

(ii) If \(P(u)\), then \(\#_a(au) = 1 + \#_a(u)\),

\[> \#_a(u)\]
Example [CST 2009, Paper2, Question 5]

$T \subseteq \{a, b\}^*$ inductively defined by

\[
\begin{array}{c}
\text{a} \quad \text{u} \\
\text{au} \\
\text{bu} \\
\text{uv}
\end{array}
\]

\[
P(u) = \#_a(u) > \#_b(u)
\]

(0) $P(a)$ holds ($1 > 0$)

(1) If $P(u)$, then $\#_a(au) = 1 + \#_a(u) > \#_a(u) > \#_b(u)$
Example [CST 2009, Paper2, Question 5]

$I \subseteq \{a,b\}^*$ inductively defined by

\[\begin{array}{ccc}
  a & \text{u} & \text{v} \\
  a & \text{au} & \text{buv} \\
\end{array}\]

\[P(u) = \#_a(u) > \#_b(u)\]

(0) \(P(a)\) holds (1 > 0)

(1) If \(P(u)\), then \(\#_a(au) = 1 + \#_a(u) > \#_a(u) > \#_b(u) = \#_b(au)\)
Example [CST 2009, Paper2, Question 5]

\( I \subseteq \{a, b\}^* \) inductively defined by

\[
\begin{array}{c c c c}
& a & u & u & v \\
\hline
a & a & au & buv
\end{array}
\]

\[ P(u) = \#_a(u) > \#_b(u) \]

(2) Suppose \( P(u) \) & \( P(v) \) hold. Then

\[ \#_a(buv) = \#_a(u) + \#_a(v) \]
Example [CST 2009, Paper 2, Question 5]

$I \subseteq \{a,b\}^*$ inductively defined by

\[
\begin{array}{c}
\begin{array}{c}
a \circ \quad u \quad a \nu \quad b \nu
\end{array}
\end{array}
\]

\[P(u) = \#_a(u) > \#_b(u)\]

(2) Suppose \(P(u)\) & \(P(v)\) hold. Then

\[\#_a(buv) = \#_a(u) + \#_a(v)\]

\[\geq (\#_b(u)+1) + (\#_b(v)+1)\]
Example [CST 2009, Paper2, Question 5]

$I \subseteq \{a, b\}^*$ inductively defined by

\[
\begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{au} \\
\text{buv}
\end{array}
\end{array}
\]

\[P(u) = \#_a(u) > \#_b(u)\]

(2) Suppose $P(u)$ & $P(v)$ hold. Then

\[
\#_a(buv) = \#_a(u) + \#_a(v)
\]

\[
\geq \#_b(u) + \#_b(v) + 2
\]

\[
> \#_b(u) + \#_b(v) + 1
\]
Example [CST 2009, Paper2, Question 5]

$\mathcal{I} \subseteq \{a, b\}^*$ inductively defined by

$$
\begin{array}{c c c}
\text{a} & \text{u} & \text{uv} \\
\text{au} & & \\
\text{bu} & \text{v} & \\
\end{array}
$$

$$P(u) = \#_a(u) > \#_b(u)$$

(2) Suppose $P(u)$ & $P(v)$ hold. Then

$$\#_a(buv) = \#_a(u) + \#_a(v)$$

$$> \#_b(u) + \#_b(v) + 1$$

$$= \#_b(buv)$$
Example [CST 2009, Paper2, Question 5]

I \subseteq \{a, b\}^* \text{ inductively defined by}

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>u,</th>
<th>u, v</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>au</td>
<td>buv</td>
</tr>
</tbody>
</table>

\[
P(u) = \#_a(u) > \#_b(u)
\]

(2) Suppose P(u) & P(v) hold. Then

\[
\#_a(buv) = \#_a(u) + \#_a(v) \\
> \#_b(u) + \#_b(v) + 1 \\
= \#_b(buv) \text{ so } P(buv) \text{ holds.}
\]
Example [CST 2009, Paper 2, Question 5]

$I \subseteq \{a, b\}^*$ inductively defined by

\[
\begin{array}{c}
\quad & u & u \vee \vspace{1em} \\
\hline
a & au & buv
\end{array}
\]

\[P(u) = \#_a(u) > \#_b(u)\]

N.B. although we have

\[\forall u \in I. \ P(u)\]

don't have \[\forall u \in \{a, b\}^*. \ P(u) \Rightarrow u \in I\]

E.g. \[P(aab), \text{ but } aab \notin I [WHY?]\]
Deciding membership of an inductively defined subset can be hard!

E.g. . . .
Collatz Conjecture

\[ f(n) = \begin{cases} 
1 & \text{if } n = 0, 1 \\
\frac{f(n)}{2} & \text{if } n \geq 1, n \text{ even} \\
3n + 1 & \text{if } n \geq 1, n \text{ odd} 
\end{cases} \]

Does this define a total function \( f : \mathbb{N} \rightarrow \mathbb{N} \)? (nobody knows)

(If it does, then \( f \) is necessarily the constantly 1 function \( n \mapsto 1 \).)
Collatz Conjecture

\[
f(n) = \begin{cases} 
1 & \text{if } n = 0, 1 \\
f(n/2) & \text{if } n > 1, n \text{ even} \\
f(3n+1) & \text{if } n > 1, n \text{ odd}
\end{cases}
\]

Does this define a total function \( f: \mathbb{N} \to \mathbb{N} \)? (nobody knows)

Can reformulate as a problem about inductively defined subsets...
Collatz Conjecture

\[ f(n) = \begin{cases} 
1 & \text{if } n = 0, 1 \\
 f(n/2) & \text{if } n > 1, n \text{ even} \\
f(3n+1) & \text{if } n > 1, n \text{ odd}
\end{cases} \]

Is the subset \( I \subseteq \mathbb{N} \) inductively defined by:

\[
\begin{array}{cccc}
0 & 1 & k & 6k+4 \\
& & 2k & 2k+1
\end{array}
\]

\( (k \geq 1) \)

equal to the whole of \( \mathbb{N} \)?