On Enumerability

Marcelo Fiore

Definition 1. A set A is said to be *enumerable* if there exists a surjection $\mathbb{N} \to A$.

Example 2. The set of integers $\mathbb{Z}$ is enumerable because there exists a surjective function $e : \mathbb{N} \to \mathbb{Z}$; take for instance the function defined as

$$e(n) = (-1)^{n \text{ mod } 2} \; ((n+1) \text{ div } 2)$$

for all $n \in \mathbb{N}$.

Example 3. The set of strings $\{0,1\}^*$ is enumerable because there exists a surjective function $e : \mathbb{N} \to \{0,1\}^*$; take for instance the function defined as

$$e(n) = b_\ell \dots b_0 \quad , \text{where } n + 1 = 2^{\ell+1} + \textstyle\sum_{k=0}^{\ell} b_k 2^k \;\; (0 \leq b_k \leq 1, \forall\, 0 \leq k \leq \ell)$$

for all $n \in \mathbb{N}$.

Lemma 4. If $f : A \to B$ and $g : B \to C$ are surjections then $g \circ f : A \to C$ is a surjection.

Corollary 5. If A is enumerable and there exists a surjection $A \to B$ then B is enumerable.

Definition 6. We let

$$\mathcal{P}_{\text{fin}}(A) = \{ S \subseteq A \mid \; S \text{ is finite } \} \quad .$$

Example 7. The function $e : \{0,1\}^* \to \mathcal{P}_{\text{fin}}(\mathbb{N})$ defined, for all $b_\ell \dots b_0 \in \{0,1\}^*$, by

$$e(b_\ell \dots b_0) = \{ i \in \mathbb{N} \mid b_i = 1 \; (0 \leq i \leq \ell) \}$$

is surjective. Hence, by Example 3 and Corollary 5, $\mathcal{P}_{\text{fin}}(\mathbb{N})$ is enumerable.

One can also show that $\mathcal{P}_{\text{fin}}(\mathbb{N})$ is enumerable directly. Indeed, the function $e : \mathbb{N} \to \mathcal{P}_{\text{fin}}(\mathbb{N})$ defined, for all $n \in \mathbb{N}$, by

$$e(n) = \{ i \in \mathbb{N} \mid b_i = 1 \text{ where } n = \sum_{k \in \mathbb{N}} b_k 2^k \;\; (0 \leq b_k \leq 1, \forall k \in \mathbb{N}) \}$$

is surjective.

Lemma 8. If A is enumerable then $\mathcal{P}_{\text{fin}}(A)$ is enumerable.

PROOF: Let $e : \mathbb{N} \to A$ be a surjection.

The function $e^{\#} : \mathcal{P}_{\text{fin}}(\mathbb{N}) \to \mathcal{P}_{\text{fin}}(A)$ defined, for all finite $S \subseteq \mathbb{N}$, by

$$e^{\#}(S) = \{e(n) \mid n \in S\}$$

is a surjection; and, by Example 7 and Corollary 5, we have that $\mathcal{P}_{\text{fin}}(A)$ is enumerable.□

Example 9. The cartesian product $\mathbb{N} \times \mathbb{N}$ is enumerable because there exists a surjection $e : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$; take for instance the function defined, for all $n \in \mathbb{N}$, by

$$e(n) = (k, \ell - k) \quad , \text{where } n = \tfrac{\ell(\ell+1)}{2} + k \text{ with } 0 \leq k \leq \ell \quad .$$

(Notice that e is surjective because, for every $(i, j) \in \mathbb{N} \times \mathbb{N}$, there exists $n = \frac{(j+i)(j+i+1)}{2} + i \in \mathbb{N}$ such that $e(n) = (i, j)$.)

Lemma 10. If A and B are enumerable then $A \times B$ is enumerable.

In particular, if A is enumerable then, for all $n \in \mathbb{N}$, A^n is enumerable.

PROOF: Let $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$ be surjections.

The function $f \times g : \mathbb{N} \times \mathbb{N} \to A \times B$ defined, for all $(m, n) \in \mathbb{N}$, by

$$(f \times g)(m, n) = (f(m), g(n))$$

is a surjection; and, by Example 9 and Corollary 5, we have that $A \times B$ is enumerable.□

Definition 11. For a set I and a family of sets $\{A_i\}_{i \in I}$ we let

$$\biguplus_{i \in I} A_i = \{(i, a) \mid \; i \in I \text{ and } a \in A_i \;\} \quad .$$

Lemma 12. For an enumerable set I and a family of enumerable sets $\{A_i\}_{i \in I}$, the set $\biguplus_{i \in I} A_i$ is enumerable.

PROOF: Let $e : \mathbb{N} \to I$ be a surjection and, for all $i \in I$, let $e_i : \mathbb{N} \to A_i$ be surjections.

The function $\varepsilon : \mathbb{N} \times \mathbb{N} \to \biguplus_{i \in I} A_i$ defined, for all $(m, n) \in \mathbb{N} \times \mathbb{N}$, by

$$\varepsilon(m, n) = (i, e_i(n)) \quad , \text{where } i = e(m)$$

is a surjection; and hence, by Example 9 and Corollary 5, we have that $\biguplus_{i \in I} A_i$ is enumerable. □

Corollary 13. If A is enumerable then A^* is enumerable.

PROOF: The function $e : \biguplus_{n \in \mathbb{N}} A^n \to A^*$ defined, for all $\ell \in \mathbb{N}$ and $a_1, \ldots, a_\ell \in A$, by

$$e(\ell, (a_1, \ldots, a_\ell)) = a_1 \ldots a_\ell$$

is surjective; and hence, by Lemma 12 and Corollary 5, we have that A^* is enumerable.□

Lemma 14. For $S \subseteq A$, if S is non-empty and A is enumerable then S is enumerable.

PROOF: Let $\emptyset \neq S \subseteq A$, and let $e : \mathbb{N} \to A$ be surjective.

Define $m : \mathbb{N} \to \mathbb{N}$ by induction as follows

$$\begin{aligned} m(0) &= \min\{n \mid e(n) \in S\} \\ m(k+1) &= \min\{n \mid n > m(k) \text{ and } e(n) \in S\} \qquad (k \in \mathbb{N}) \end{aligned}$$

where, by convention, $\min \emptyset = m(0)$.

Then, the function $e' : \mathbb{N} \to S$ defined, for all $k \in \mathbb{N}$, by

$$e'(k) = e(m(k))$$

is surjective; and hence S is enumerable. □

Corollary 15. Non-empty finite sets are enumerable.