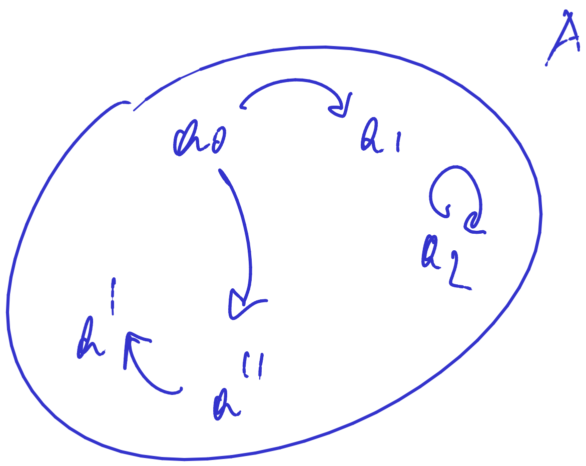
Directed graphs

Definition 104 *A* <u>directed graph</u> (A, R) *consists of a set* A *and a relation* R *on* A *(i.e. a relation from* A *to* A*).*



$(a_0, a_1) \in R$

$(a_2, a_2) \in R$

$\vdots$

Corollary 106 *For every set* A*, the structure*

$$\left(\mathrm{Rel}(A)\ ,\ \mathrm{id}_A\ ,\circ\ \right)$$

the set of all relations from A to A; i.e. the set of all directed graphs on A.

$$\mathrm{Rel}(A) = \mathcal{P}(A \times A)$$

is a monoid.

Definition 107 *For* $\underline{R \in \mathrm{Rel}(A)}$ *and* $n \in \mathbb{N}$*, we let*

$$R^{\circ n} = \underbrace{R \circ \cdots \circ R}_{n \text{ times}} \in \mathrm{Rel}(A)$$

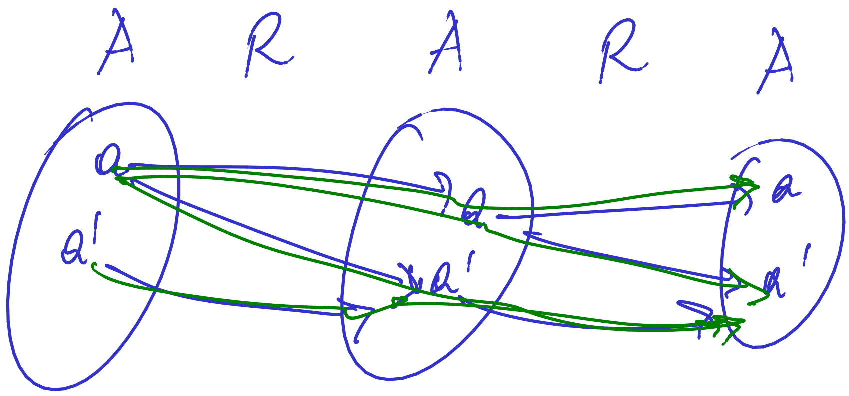
be defined as id_A *for* $n = 0$*, and as* $R \circ R^{\circ m}$ *for* $n = m + 1$*.*

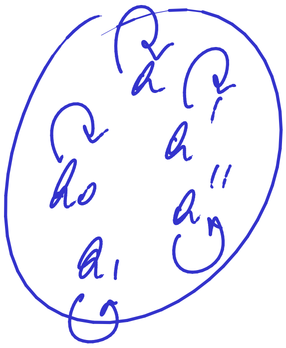
Paths

Proposition 109 *Let* (A, R) *be a directed graph. For all* $n \in \mathbb{N}$ *and* $s, t \in A$, $s \; R^{\circ n} \; t$ *iff there exists a path of length* n *in* R *with source* s *and target* t.

$R^{\circ 0} = \mathrm{id}_A$

PROOF: $R \circ R$





R^2

PROOF idea Show $R^{\circ n}$ relates two elements iff there is a path of length n in the directed graph by induction on n.

Inductive step. We need the above for $R^{\circ n+1}$

Assume by (I.H) That it is the case for $R^{\circ n}$ and consider the definition of

$$R^{\circ n} \circ R$$

notation for $\bigcup \mathcal{F}$, $\mathcal{F} = \{ X \in \mathrm{Rel}(A) \mid \exists n \in \mathbb{N}.\ X = R^{\circ n} \}$

Definition 110 *For* $R \in \mathrm{Rel}(A)$, *let*

$$R^{\circ *} \;=\; \bigcup \left\{ R^{\circ n} \in \mathrm{Rel}(A) \mid n \in \mathbb{N} \right\} \;=\; \bigcup_{n \in \mathbb{N}} R^{\circ n} \quad .$$

Corollary 111 *Let* (A, R) *be a directed graph. For all* $s, t \in A$, $s \; R^{\circ *} \; t$ *iff there exists a path with source* s *and target* t *in* R.

$\underline{\mathrm{mat}}(R \circ S) = \underline{\mathrm{mat}}(R) \bullet \mathrm{mat}(S)$

$\underline{\mathrm{mat}}(R \cup S) = \underline{\mathrm{mat}}(R) + \underline{\mathrm{mat}}(S)$

The $(n \times n)$-matrix $M = \mathrm{mat}(R)$ of a finite directed graph $([n], R)$ for n a positive integer is called its *adjacency matrix*.

The adjacency matrix $M^* = \mathrm{mat}(R^{\circ *})$ can be computed by matrix multiplication and addition as M_n where

$$\begin{cases} M_0 & = & I_n \\ M_{k+1} & = & I_n + (M \cdot M_k) \end{cases}$$

This gives an algorithm for establishing or refuting the existence of paths in finite directed graphs.

Preorders

$A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$

Definition 112 *A* preorder $(\, P \,,\, \sqsubseteq \,)$ *consists of a set* P *and a relation* $\sqsubseteq$ *on* P *(i.e.* $\sqsubseteq \, \in \mathcal{P}(P \times P)$*) satisfying the following two axioms.*

- *Reflexivity.*

$$\forall x \in P.\ x \sqsubseteq x$$

- *Transitivity.*

$$\forall x, y, z \in P.\ (x \sqsubseteq y \ \wedge \ y \sqsubseteq z) \implies x \sqsubseteq z$$

Preorders with an additional property

$x \preceq y \wedge y \preceq x \Rightarrow x = y$

Anti-symmetry.

leading to the notion of partial order

Examples:

- $(\mathbb{R}, \leq)$ and $(\mathbb{R}, \geq)$.
- $(\mathcal{P}(A), \subseteq)$ and $(\mathcal{P}(A), \supseteq)$.
- $(\mathbb{Z}, \mid)$.

$m \mid n \wedge n \mid k \Rightarrow m \mid k$

$n \mid n$

Not a partial order

$2 \mid -2 \wedge -2 \mid 2$

but $2 \neq -2$

Theorem 114 *For* $R \subseteq A \times A$, *let*

$$\mathcal{F}_R = \{ Q \subseteq A \times A \mid R \subseteq Q \wedge Q \textit{ is a preorder} \} .$$

Then, (i) $R^{\circ *} \in \mathcal{F}_R$ *and* (ii) $R^{\circ *} \subseteq \bigcap \mathcal{F}_R$. *Hence,* $R^{\circ *} = \bigcap \mathcal{F}_R$.

PROOF:

$R \subseteq R^{\circ *}$

$\wedge$ $R^{\circ *}$ is a preorder. reflexivity. transitivity.

$x \, R^{\circ *} \, y \iff$ There is a path, say of length k, from x to y

$y \, R^{\circ *} \, z \iff$ there is a path, say of length l, from y to z.

$$R^{\circ *} \subseteq \bigcap \mathcal{F}_R$$

$$= \bigcup_{n\in\mathbb{N}} R^{\circ n}$$

$Y \subseteq \bigcap_j B_j$

iff $Y \subseteq B_j \ \forall j$

$$\bigcup_{n\in\mathbb{N}} R^{\circ n} \subseteq \bigcap \mathcal{F}_R$$

iff $\forall n\in\mathbb{N}.\ R^{\circ n} \subseteq \bigcap \mathcal{F}_R$

$\bigcup_i A_i \subseteq X$

iff $\forall i.\ A_i \subseteq X$

iff $\forall Q \in \mathcal{F}_R.\ \forall n \in \mathbb{N}.\ R^{\circ n} \subseteq Q$

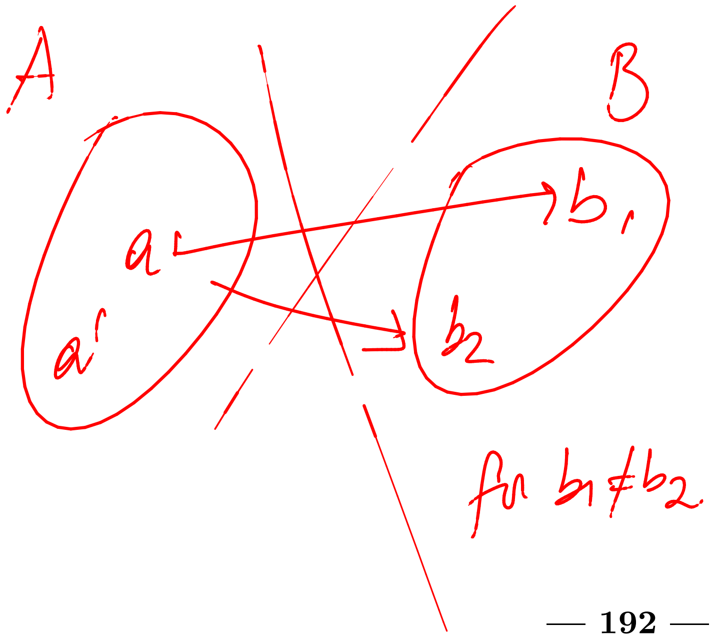
by induction

Hence there is a path from x to z, of length $k+l$ (obtaining by concatenating the paths from x to y and from y to z) and so $x \, R^{\circ *} \, z$.

Partial functions

Definition 115 *A relation* $R : A \rightarrowtriangle B$ *is said to be* functional, *and called a* partial function, *whenever it is such that*

$$\forall a \in A. \forall b_1, b_2 \in B.\ a\,R\,b_1 \wedge a\,R\,b_2 \implies b_1 = b_2 \quad .$$



Since every $a \in A$ is related then it is so to a unique $b \in B$, we give it a name

$R(a)$

Theorem 117 *The identity relation is a partial function, and the composition of partial functions yields a partial function.*

NB

$$f = g : A \rightharpoonup B$$

iff

$$\forall a \in A.\, \big(f(a){\downarrow} \iff g(a){\downarrow} \big) \wedge f(a) = g(a)$$

<u>Notation</u> For a partial function $f : A \rightharpoonup B$
$f(a)\uparrow$ ~ f is undefined for a.
$f(a)\downarrow$ ~ f is defined for a.

Proposition 118 *For all finite sets* A *and* B,

$$\#\underbrace{(A \rightharpoonup B)} = (\#B+1)^{\#A} \ .$$

the set of all partial functions from A to B.

PROOF IDEA:

$A = \{a_1 \ldots a_m\}$

$B = \{b_1 \ldots b_n\}$

$(A \rightharpoonup B) \subseteq \mathcal{P}(A \times B)$

$a_1 \to b_1, b_2, \ldots, b_n$: $n+1$

$a_2 \to b_1, b_2, \ldots, b_n$: $n+1$

$\ldots$

$(n+1) \times (n+1) \times \cdots \times (n+1) = (n+1)^m$

$$\#\mathcal{P}(A \times B) = 2^{\#(A\times B)} = 2^{\#A \cdot \#B}$$