Powerset axiom

For any set, there is a set consisting of all its subsets.

\[ \forall X. \, X \in \mathcal{P}(U) \iff X \subseteq U. \]
The powerset Boolean algebra

\[( \mathcal{P}(U), \emptyset, U, \cup, \cap, (\cdot)^c ) \]

For all \( A, B \in \mathcal{P}(U) \),

\[
A \cup B = \{ x \in U \mid x \in A \lor x \in B \} \in \mathcal{P}(U)
\]

\[
A \cap B = \{ x \in U \mid x \in A \land x \in B \} \in \mathcal{P}(U)
\]

\[
A^c = \{ x \in U \mid \neg(x \in A) \} \in \mathcal{P}(U)
\]
The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

\[
(A \cup B) \cup C = A \cup (B \cup C), \quad A \cup B = B \cup A, \quad A \cup A = A
\]

\[
(A \cap B) \cap C = A \cap (B \cap C), \quad A \cap B = B \cap A, \quad A \cap A = A
\]
The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C) \ , \ A \cup B = B \cup A \ , \ A \cup A = A$$

$$(A \cap B) \cap C = A \cap (B \cap C) \ , \ A \cap B = B \cap A \ , \ A \cap A = A$$

The empty set $\emptyset$ is a neutral element for $\cup$ and the universal set $U$ is a neutral element for $\cap$.

$$\emptyset \cup A = A = U \cap A$$
The empty set $\emptyset$ is an annihilator for $\cap$ and the universal set $U$ is an annihilator for $\cup$.

$$\emptyset \cap A = \emptyset$$

$$U \cup A = U$$
The empty set $\emptyset$ is an annihilator for $\cap$ and the universal set $U$ is an annihilator for $\cup$.

$$\emptyset \cap A = \emptyset$$

$$U \cup A = U$$

With respect to each other, the union operation $\cup$ and the intersection operation $\cap$ are distributive and absorptive.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) , \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cup (A \cap B) = A = A \cap (A \cup B)$$
The complement operation \((\cdot)^c\) satisfies complementation laws.

\[
A \cup A^c = U, \quad A \cap A^c = \emptyset
\]
Sets and logic

<table>
<thead>
<tr>
<th>$\mathcal{P}(\mathcal{U})$</th>
<th>${ \text{false, true} }$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>false</td>
</tr>
<tr>
<td>$\mathcal{U}$</td>
<td>true</td>
</tr>
<tr>
<td>$\cup$</td>
<td>$\lor$</td>
</tr>
<tr>
<td>$\cap$</td>
<td>$\land$</td>
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<tr>
<td>$(\cdot)^c$</td>
<td>$\lnot(\cdot)$</td>
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Pairing axiom

For every $a$ and $b$, there is a set with $a$ and $b$ as its only elements.

$$\{a, b\}$$

defined by

$$\forall x. x \in \{a, b\} \iff (x = a \lor x = b)$$

NB The set $\{a, a\}$ is abbreviated as $\{a\}$, and referred to as a singleton.
Examples:

- $\# \{ \emptyset \} = 1$
- $\# \{ \{ \emptyset \} \} = 1$
- $\# \{ \emptyset, \{ \emptyset \} \} = 2$
Ordered pairing

For every pair $a$ and $b$, the set

$$\{ \{a\}, \{a, b\} \}$$

is abbreviated as

$$\langle a, b \rangle$$

and referred to as an ordered pair.
Proposition 83 (Fundamental property of ordered pairing)

For all $a, b, x, y$,

$$\langle a, b \rangle = \langle x, y \rangle \iff (a = x \land b = y) .$$

Proof:
Products

The **product** $A \times B$ of two sets $A$ and $B$ is the set

$$A \times B = \{ x \mid \exists a \in A, b \in B. x = (a, b) \}$$

where

$$\forall a_1, a_2 \in A, b_1, b_2 \in B. \quad (a_1, b_1) = (a_2, b_2) \iff (a_1 = a_2 \land b_1 = b_2) .$$

Thus,

$$\forall x \in A \times B. \exists! a \in A. \exists! b \in B. x = (a, b) .$$
Proposition 85  For all finite sets $A$ and $B$,

$$\#(A \times B) = \#A \cdot \#B.$$

Proof idea:
**Big unions**

**Definition 86** Let $U$ be a set. For a collection of sets $F \in \mathcal{P}(\mathcal{P}(U))$, we let the **big union** (relative to $U$) be defined as

$$\bigcup F = \{ x \in U \mid \exists A \in F. x \in A \} \in \mathcal{P}(U).$$