Natural numbers

In the beginning there were the *natural numbers*

$$\mathbb{N} \; : \quad 0\;, \quad 1\;, \quad \ldots\;, \quad n\;, \quad n+1\;, \quad \ldots$$

generated from *zero* by successive increment; that is, put in ML:

```
datatype
  N = zero | succ of N
```

The basic operations of this number system are:

- Addition

$$\underbrace{\overbrace{* \cdots *}^{m}\,\overbrace{* \cdots \cdots *}^{n}}_{m+n}$$

- Multiplication

$$m \left\{ \begin{array}{ccc} \multicolumn{3}{c}{\overbrace{* \cdots \cdots \cdots \cdots *}^{n}} \\ \vdots & m \cdot n & \vdots \\ \multicolumn{3}{c}{* \cdots \cdots \cdots \cdots *} \end{array} \right.$$

The *additive structure* $(\mathbb{N}, 0, +)$ of natural numbers with zero and addition satisfies the following:

- Monoid laws

$$0 + n = n = n + 0 \quad , \quad (l + m) + n = l + (m + n)$$

- Commutativity law

$$m + n = n + m$$

and as such is what in the mathematical jargon is referred to as a *commutative monoid*.

Also the *multiplicative structure* $(\mathbb{N}, 1, \cdot)$ of natural numbers with one and multiplication is a commutative monoid:

- Monoid laws

$$1 \cdot n = n = n \cdot 1 \quad , \quad (l \cdot m) \cdot n = l \cdot (m \cdot n)$$

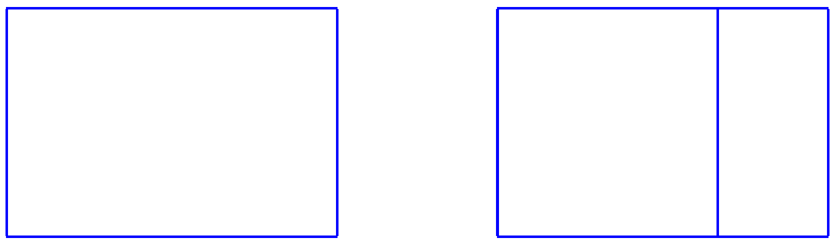
- Commutativity law

$$m \cdot n = n \cdot m$$

The additive and multiplicative structures interact nicely in that they satisfy the

- Distributive law

$$l \cdot (m + n) \;=\; l \cdot m + l \cdot n$$



and make the overall structure $(\mathbb{N}, 0, +, 1, \cdot)$ into what in the mathematical jargon is referred to as a *commutative semiring*.

Cancellation

The additive and multiplicative structures of natural numbers further satisfy the following laws.

- Additive cancellation

  For all natural numbers k, m, n,

  $$k + m = k + n \implies m = n \quad .$$

- Multiplicative cancellation

  For all natural numbers k, m, n,

  if $k \neq 0$ then $k \cdot m = k \cdot n \implies m = n$.

Inverses

Definition 41

1. *A number* x *is said to admit an* additive inverse *whenever there exists a number* y *such that* $x + y = 0$.

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1. *A number x is said to admit an* additive inverse *whenever there exists a number y such that $x + y = 0$.*

2. *A number x is said to admit a* multiplicative inverse *whenever there exists a number y such that $x \cdot y = 1$.*

Extending the system of natural numbers to: (i) admit all additive inverses and then (ii) also admit all multiplicative inverses for non-zero numbers yields two very interesting results:

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(i) the *integers*

$$\mathbb{Z} \ : \quad \ldots -n\,,\ \ldots\,,\ -1\,,\ 0\,,\ 1\,,\ \ldots\,,\ n\,,\ \ldots$$

which then form what in the mathematical jargon is referred to as a *commutative ring*, and

(ii) the *rationals* $\mathbb{Q}$ which then form what in the mathematical jargon is referred to as a *field*.

The division theorem and algorithm

Theorem 42 (Division Theorem) *For every natural number* m *and positive natural number* n*, there exists a unique pair of integers* q *and* r *such that* $q \geq 0$, $0 \leq r < n$, *and* $m = q \cdot n + r$.

The division theorem and algorithm

Theorem 42 (Division Theorem) *For every natural number* m *and positive natural number* n*, there exists a unique pair of integers* q *and* r *such that* $q \geq 0$*,* $0 \leq r < n$*, and* $m = q \cdot n + r$*.*

Definition 43 *The natural numbers* q *and* r *associated to a given pair of a natural number* m *and a positive integer* n *determined by the Division Theorem are respectively denoted* $\mathrm{quo}(m, n)$ *and* $\mathrm{rem}(m, n)$*.*

The Division Algorithm in ML:

```
fun divalg( m , n )
  = let
      fun diviter( q , r )
        = if r < n then ( q , r )
          else diviter( q+1 , r-n )
    in
      diviter( 0 , m )
    end

fun quo( m , n ) = #1( divalg( m , n ) )

fun rem( m , n ) = #2( divalg( m , n ) )
```

Theorem 44 *For every natural number* m *and positive natural number* n*, the evaluation of* $\texttt{divalg}(m, n)$ *terminates, outputing a pair of natural numbers* (q_0, r_0) *such that* $r_0 < n$ *and* $m = q_0 \cdot n + r_0$*.*

PROOF: