The main proof strategy for implication:

To prove a goal of the form

$$P \implies Q$$

assume that P is true and prove Q.

NB *Assuming* is not *asserting*! Assuming a statement amounts to the same thing as adding it to your list of hypotheses.

An alternative proof strategy for implication:

To prove an implication, prove instead the equivalent statement given by its contrapositive.

Definition:

the *contrapositive* of 'P implies Q' is 'not Q implies not P'

Proof pattern:

In order to prove that

$$\mathrm{P} \implies \mathrm{Q}$$

1. Write: We prove the contrapositive; that is, ... and state the contrapositive.
2. Write: Assume 'the negation of Q'.
3. Show that 'the negation of P' logically follows.

Scratch work:

Before using the strategy

Assumptions	Goal
	$P \implies Q$
$\vdots$	

After using the strategy

Assumptions	Goal
	not P
$\vdots$	
not Q	

Definition 9 *A real number is:*

- rational *if it is of the form m/n for a pair of integers m and n; otherwise it is* irrational*.*
- positive *if it is greater than 0, and* negative *if it is smaller than 0.*
- nonnegative *if it is greater than or equal 0, and* nonpositive *if it is smaller than or equal 0.*
- natural *if it is a nonnegative integer.*

Proposition 10 *Let x be a positive real number. If x is irrational then so is $\sqrt{x}$.*

PROOF:

Logical Deduction — Modus Ponens —

A main rule of *logical deduction* is that of *Modus Ponens*:

> From the statements P and $P \implies Q$,
> the statement Q follows.

or, in other words,

> If P and $P \implies Q$ hold then so does Q.

or, in symbols,

$$\frac{P \qquad P \implies Q}{Q}$$

The use of implications:

To use an assumption of the form $P \implies Q$, aim at establishing P.
Once this is done, by Modus Ponens, one can conclude Q and so further assume it.

Theorem 11 *Let* P_1, P_2, *and* P_3 *be statements. If* $P_1 \implies P_2$ *and* $P_2 \implies P_3$ *then* $P_1 \implies P_3$.

PROOF:

Bi-implication

Some theorems can be written in the form

$$\boxed{\text{P is equivalent to Q}}$$

or, in other words,

$$\boxed{\text{P implies Q, and vice versa}}$$

or

$$\boxed{\text{Q implies P, and vice versa}}$$

or

$$\boxed{\text{P if, and only if, Q}} \qquad \boxed{\text{P iff Q}}$$

or, in symbols,

$$\boxed{P \iff Q}$$

Proof pattern:

In order to prove that

$$P \iff Q$$

1. Write: $(\Longrightarrow)$ and give a proof of $P \implies Q$.
2. Write: $(\Longleftarrow)$ and give a proof of $Q \implies P$.

Proposition 12 *Suppose that n is an integer. Then, n is even iff n^2 is even.*

PROOF:

Divisibility and congruence

Definition 13 *Let d and n be integers. We say that d <u>divides</u> n, and write $d \mid n$, whenever there is an integer k such that $n = k \cdot d$.*

Example 14 *The statement $2 \mid 4$ is true, while $4 \mid 2$ is not.*

Definition 15 *Fix a positive integer m. For integers a and b, we say that a is congruent to b modulo m, and write $a \equiv b \pmod{m}$, whenever $m \mid (a - b)$.*

Example 16

1. $18 \equiv 2 \pmod{4}$
2. $2 \equiv -2 \pmod{4}$
3. $18 \equiv -2 \pmod{4}$

Proposition 17 *For every integer n,*

1. n *is even if, and only if,* $n \equiv 0 \pmod{2}$, *and*
2. n *is odd if, and only if,* $n \equiv 1 \pmod{2}$.

PROOF:

The use of bi-implications:

To use an assumption of the form $P \iff Q$, use it as two separate assumptions $P \implies Q$ and $Q \implies P$.