The main proof strategy for implication:

To prove a goal of the form

 $P \implies Q$

assume that P is true and prove Q.

NB Assuming is not asserting! Assuming a statement amounts to the same thing as adding it to your list of hypotheses.

An alternative proof strategy for implication:

To prove an implication, prove instead the equivalent statement given by its contrapositive.

Definition:

the *contrapositive* of 'P implies Q' is 'not Q implies not P'

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Proof pattern:

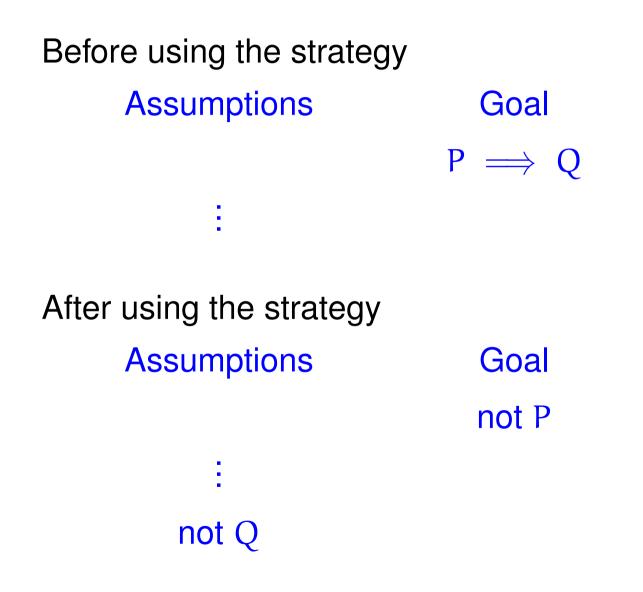
In order to prove that

$P \implies Q$

1. Write: We prove the contrapositive; that is, ... and state the contrapositive.

- **2.** Write: Assume 'the negation of Q'.
- 3. Show that 'the negation of P' logically follows.

Scratch work:



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Definition 9 A real number is:

- rational if it is of the form m/n for a pair of integers m and n; otherwise it is irrational.
- ▶ positive if it is greater than 0, and negative if it is smaller than 0.
- nonnegative if it is greater than or equal 0, and nonpositive if it is smaller than or equal 0.

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▶ <u>natural</u> if it is a nonnegative integer.

Proposition 10 Let x be a positive real number. If x is irrational

then so is \sqrt{x} .

Logical Deduction – Modus Ponens –

A main rule of *logical deduction* is that of *Modus Ponens*:

From the statements P and P \implies Q, the statement Q follows.

or, in other words,

If P and P \implies Q hold then so does Q.

or, in symbols,

$$\begin{array}{ccc} P & P \implies Q \\ \hline Q \end{array}$$

The use of implications:

To use an assumption of the form $P \implies Q$, aim at establishing P. Once this is done, by Modus Ponens, one can conclude Q and so further assume it.

Theorem 11 Let P_1 , P_2 , and P_3 be statements. If $P_1 \implies P_2$ and $P_2 \implies P_3$ then $P_1 \implies P_3$.

Bi-implication

Some theorems can be written in the form

P is equivalent to Q

or, in other words,

P implies Q, and vice versa

or

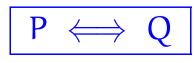
Q implies P, and vice versa

or

P if, and only if, Q

P iff Q

or, in symbols,



Proof pattern:

In order to prove that

$$\mathsf{P} \iff \mathsf{Q}$$

— **3**4 —

1. Write: (\Longrightarrow) and give a proof of $P \implies Q$.

2. Write: (\iff) and give a proof of $Q \implies P$.

Proposition 12 Suppose that n is an integer. Then, n is even iff n^2 is even.

Divisibility and congruence

Definition 13 Let d and n be integers. We say that d divides n, and write $d \mid n$, whenever there is an integer k such that $n = k \cdot d$.

Example 14 The statement 2 | 4 is true, while 4 | 2 is not.

Definition 15 Fix a positive integer m. For integers a and b, we say that a is congruent to b modulo m, and write $a \equiv b \pmod{m}$, whenever $m \mid (a - b)$.

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Example 16

- **1.** $18 \equiv 2 \pmod{4}$
- **2.** $2 \equiv -2 \pmod{4}$
- *3.* $18 \equiv -2 \pmod{4}$

Proposition 17 For every integer n,

1. n is even if, and only if, $n \equiv 0 \pmod{2}$, and

2. n is odd if, and only if, $n \equiv 1 \pmod{2}$.

The use of bi-implications:

To use an assumption of the form P \iff Q, use it as two separate assumptions P \implies Q and Q \implies P.