Denotational semantics of PCF

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$*, the denotation*

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

is a well-defined continous function.

Denotations of closed terms

For a closed term $M \in \mathrm{PCF}_\tau$, we get

$$[\![\emptyset \vdash M]\!] : [\![\emptyset]\!] \to [\![\tau]\!]$$

and, since $[\![\emptyset]\!] = \{\perp\}$, we have

$$\boxed{[\![M]\!] \stackrel{\text{def}}{=} [\![\emptyset \vdash M]\!](\perp) \in [\![\tau]\!] \qquad (M \in \mathrm{PCF}_\tau)}$$

Proof By induction.

Compositionality

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$ *and* $\Gamma \vdash M' : \tau$, *and all contexts* $\mathcal{C}[-]$ *such that* $\Gamma' \vdash \mathcal{C}[M] : \tau'$ *and* $\Gamma' \vdash \mathcal{C}[M'] : \tau'$,

$$\textit{if } [\![\Gamma \vdash M]\!] = [\![\Gamma \vdash M']\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

$$\textit{then } [\![\Gamma' \vdash \mathcal{C}[M]]\!] = [\![\Gamma' \vdash \mathcal{C}[M]]\!] : [\![\Gamma']\!] \to [\![\tau']\!]$$

$\mathcal{C}[-] = \underline{\mathrm{fix}}[-]$

Example

$[\![\Gamma \vdash M]\!] = [\![\Gamma \vdash M']\!] : [\![\Gamma]\!] \longrightarrow ([\![\tau]\!] \to [\![\tau]\!])$

$\Rightarrow [\![\Gamma \vdash \underline{\mathrm{fix}}(M)]\!] = [\![\Gamma \vdash \underline{\mathrm{fix}}(M')]\!] : [\![\Gamma]\!] \longrightarrow [\![\tau]\!]$

$$
\begin{aligned}
[\![\Gamma \vdash fix(M)]\!] &= \underline{fix} \circ [\![\Gamma \vdash M]\!] \\
&= \underline{fix} \circ [\![\Gamma \vdash M']\!] \\
&= [\![\Gamma \vdash \underline{fix}(M')]\!]
\end{aligned}
$$

Soundness

Proposition. *For all closed terms* $M, V \in \mathrm{PCF}_\tau$,

if $M \Downarrow_\tau V$ *then* $[\![M]\!] = [\![V]\!] \in [\![\tau]\!]$.

Proof By induction on the derivation of $M \Downarrow_\tau V$.

Example

$$\frac{M_1 \Downarrow \mathrm{fn}\, x.\, M' \qquad M'[M_2/x] \Downarrow V}{M = M_1\, M_2 \Downarrow V}$$

By induction

- $[\![M_1]\!] = [\![\mathrm{fn}\, x. M']\!] = \lambda d. [\![M']\!][x \mapsto d]$

- $[\![M'[M_2/x]]\!] = [\![V]\!]$

NEED TO BE RELATED!

Then

$[\![M_1 M_2]\!] = [\![M_1]\!]([\![M_2]\!]) = [\![M']\!][x \mapsto [\![M_2]\!]]$

Substitution property

Proposition. *Suppose that* $\Gamma \vdash M : \tau$ *and that* $\Gamma[x \mapsto \tau] \vdash M' : \tau'$*, so that we also have* $\Gamma \vdash M'[M/x] : \tau'$*.*

Then,

$$\begin{aligned} & [\![\Gamma \vdash M'[M/x]]\!](\rho) \\ & \quad = [\![\Gamma[x \mapsto \tau] \vdash M']\!]\big(\rho\big[x \mapsto [\![\Gamma \vdash M]\!]\big]\big) \end{aligned}$$

for all $\rho \in [\![\Gamma]\!]$*.*

Syntactic substitution
is interpreted as function
application

Substitution property

Proposition. *Suppose that* $\Gamma \vdash M : \tau$ *and that* $\Gamma[x \mapsto \tau] \vdash M' : \tau'$, *so that we also have* $\Gamma \vdash M'[M/x] : \tau'$.

Then,

$$\begin{aligned} &[\![\Gamma \vdash M'[M/x]]\!](\rho) \\ &\quad = [\![\Gamma[x \mapsto \tau] \vdash M']\!]\big(\rho\big[x \mapsto [\![\Gamma \vdash M]\!]\big]\big) \end{aligned}$$

for all $\rho \in [\![\Gamma]\!]$.

In particular when $\Gamma = \emptyset$, $[\![\langle x \mapsto \tau\rangle \vdash M']\!] : [\![\tau]\!] \to [\![\tau']\!]$ and

$$[\![M'[M/x]]\!] = [\![\langle x \mapsto \tau\rangle \vdash M']\!]([\![M]\!])$$

Topic 7

Relating Denotational and Operational Semantics

Adequacy

For any closed PCF terms M and V of *ground* type $\gamma \in \{nat, bool\}$ with V a value

$$[\![M]\!] = [\![V]\!] \in [\![\gamma]\!] \implies M \Downarrow_\gamma V\,.$$

ground type

"observable behaviour"

Adequacy

For any closed PCF terms M and V of *ground* type $\gamma \in \{nat, bool\}$ with V a value

$$[\![M]\!] = [\![V]\!] \in [\![\gamma]\!] \implies M \Downarrow_\gamma V \, .$$

NB. Adequacy does not hold at function types:

$$[\![\mathbf{fn}\, x : \tau .\, (\mathbf{fn}\, y : \tau .\, y)\, x]\!] \quad = \quad [\![\mathbf{fn}\, x : \tau .\, x]\!] \quad : [\![\tau]\!] \to [\![\tau]\!]$$

$$= \lambda d . [\![\mathrm{fn}\, y . y]\!][x \mapsto d]$$

$$= \lambda d . [\![(\mathrm{fn}\, y . y)\, x]\!][x \mapsto d]$$

$$= \lambda d . ([\![\mathrm{fn}\, y . y]\!](x \mapsto d])([\![x]\!][x \mapsto d]) = \lambda d\, (\lambda e . e)\, d$$

$$= \lambda d . d$$

$$= \lambda d . [\![x]\!][x \mapsto d] = \lambda d . d$$

Adequacy

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$$[\![M]\!] = [\![V]\!] \in [\![\gamma]\!] \implies M \Downarrow_\gamma V\ .$$

NB. Adequacy does not hold at function types:

$$[\![\mathbf{fn}\, x : \tau .\, (\mathbf{fn}\, y : \tau .\, y)\, x]\!] \quad = \quad [\![\mathbf{fn}\, x : \tau .\, x]\!] \quad : [\![\tau]\!] \to [\![\tau]\!]$$

but

$$\mathbf{fn}\, x : \tau .\, (\mathbf{fn}\, y : \tau .\, y)\, x \not\Downarrow_{\tau \to \tau} \mathbf{fn}\, x : \tau .\, x$$

Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   ▶ Consider M to be $M_1\, M_2$, $\mathbf{fix}(M')$.

$[\![M]\!] = [\![V]\!] \in [\![\gamma]\!] \Rightarrow M \Downarrow_\gamma V$ $\quad \gamma$ ground.

say $M = M_1\, M_2$ — is of function type

$[\![M_1 M_2]\!] = [\![M_1]\!]([\![M_2]\!])$

Idea: We will prove something general for all types that implies adequacy (at ground types).

Adequacy proof idea

at ground type we define $\lhd_\gamma$ so that adequacy is implied.

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   ▶ Consider M to be $M_1 M_2$, $\mathbf{fix}(M')$.

2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.

   This statement roughly takes the form:

   $$\boxed{[\![M]\!] \lhd_\tau M \text{ for all types } \tau \text{ and all } M \in \mathrm{PCF}_\tau}$$

   the statement $[\![M]\!] \lhd_\gamma M$ will imply adequacy

   at higher type $\lhd_{\sigma \to \tau}$?

   where the *formal approximation relations*

   $$\lhd_\tau \subseteq [\![\tau]\!] \times \mathrm{PCF}_\tau$$

   are *logically* chosen to allow a proof by induction.

→ LOGICAL RELATIONS.

Requirements on the formal approximation relations, I

We want that, for $\gamma \in \{nat, bool\}$,

$$[\![M]\!] \lhd_\gamma M \text{ implies } \underbrace{\forall V\,([\![M]\!] = [\![V]\!] \implies M \Downarrow_\gamma V)}_{\text{adequacy}}$$

$\gamma = \underline{nat}$

$n \lhd_{\underline{nat}} M$?

remark

$[\![\underline{\mathbf{succ}}^n(0)]\!] = n \in \mathbb{N}$

Definition of $d \lhd_\gamma M$ $(d \in [\![\gamma]\!], M \in \mathrm{PCF}_\gamma)$

for $\gamma \in \{nat, bool\}$

$$n \lhd_{nat} M \overset{\text{def}}{\Leftrightarrow} (n \in \mathbb{N} \Rightarrow M \Downarrow_{nat} \mathbf{succ}^n(\mathbf{0}))$$

$$b \lhd_{bool} M \overset{\text{def}}{\Leftrightarrow} (b = true \Rightarrow M \Downarrow_{bool} \mathbf{true})$$
$$\& \ (b = false \Rightarrow M \Downarrow_{bool} \mathbf{false})$$

Suppose $[\![M]\!] \lhd_{nat} M \overset{?}{\Rightarrow} ([\![M]\!] = [\![V]\!] \Rightarrow M \Downarrow V)$

Assume $[\![M]\!] = [\![V]\!]$, say $V = \underline{succ}^n(0)$. Then $[\![M]\!] = n$

and $n \lhd_{nat} M \Rightarrow M \Downarrow_n succ^n(0) = V$

Proof of: $[\![M]\!] \lhd_\gamma M$ implies **adequacy**

Case $\gamma = nat$.

$$
\begin{aligned}
& [\![M]\!] = [\![V]\!] \\
& \quad \Longrightarrow \; [\![M]\!] = [\![\mathbf{succ}^n(\mathbf{0})]\!] && \text{for some } n \in \mathbb{N} \\
& \quad \Longrightarrow \; n = [\![M]\!] \lhd_\gamma M \\
& \quad \Longrightarrow \; M \Downarrow \mathbf{succ}^n(\mathbf{0}) && \text{by definition of } \lhd_{nat}
\end{aligned}
$$

Case $\gamma = bool$ is similar.

Want to prove $[\![M]\!] \lhd_\tau M$ for all τ by induction

$f \lhd_{\sigma\to\tau} M \overset{\text{def}}{\iff} \forall\, d \lhd_\sigma N.\ f(d) \lhd_\tau M\, N$

Requirements on the formal approximation relations, II

We want to be able to proceed by induction.

- Consider the case $M = M_1\, M_2$.

$$\frac{[\![M_1]\!] \lhd_{\sigma\to\tau} M_1 \qquad [\![M_2]\!] \lhd_\sigma M_2}{[\![M_1]\!]([\![M_2]\!]) = [\![M_1\, M_2]\!] \lhd_\tau M_1\, M_2}$$

? RTP

by induction

$\rightsquigarrow$ *logical* definition