Topic 3

Constructions on Domains

Example $\mathbb{N}_\perp$, $\mathbb{B}_\perp$ $\mathbb{B} = \{true, false\}$.

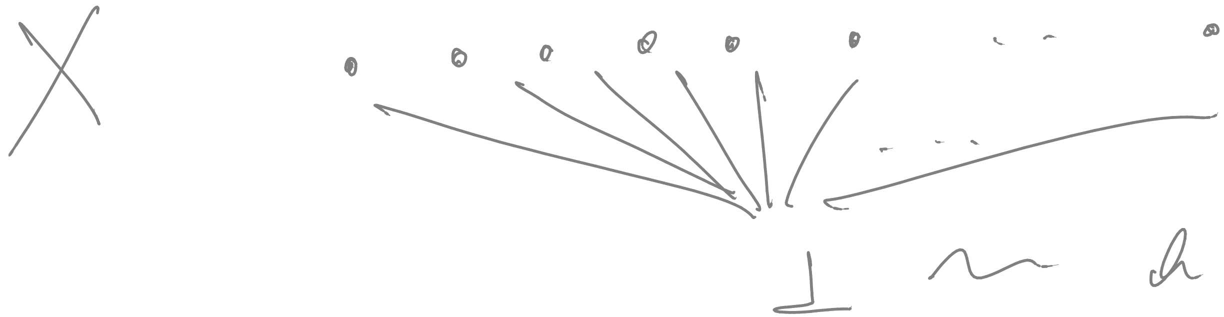
Discrete cpo's and flat domains

For any set X, the relation of equality

$$x \sqsubseteq x' \overset{\text{def}}{\Leftrightarrow} x = x' \qquad (x, x' \in X)$$

makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set X.

Every set can be made into a domain.



$\rightsquigarrow$ a new least element

Discrete cpo's and flat domains

For any set X, the relation of equality

$$x \sqsubseteq x' \stackrel{\text{def}}{\Leftrightarrow} x = x' \qquad (x, x' \in X)$$

makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set X.

Let $X_\perp \stackrel{\text{def}}{=} X \cup \{\perp\}$, where $\perp$ is some element not in X. Then

$$d \sqsubseteq d' \stackrel{\text{def}}{\Leftrightarrow} (d = d') \vee (d = \perp) \qquad (d, d' \in X_\perp)$$

makes $(X_\perp, \sqsubseteq)$ into a domain (with least element $\perp$), called the flat domain determined by X.

Binary product of cpo's and domains

The product of two cpo's $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \ \& \ d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined by

check this is a partial order

$$(d_1, d_2) \sqsubseteq (d'_1, d'_2) \overset{\text{def}}{\Leftrightarrow} d_1 \sqsubseteq_1 d'_1 \ \& \ d_2 \sqsubseteq_2 d'_2 \ .$$

Claim If D_1 and D_2 are domains then the construction $D_1 \times D_2$ is also a domain.

$$\frac{(x_1, x_2) \sqsubseteq (y_1, y_2)}{\dfrac{}{x_1 \sqsubseteq_1 y_1 \qquad x_2 \sqsubseteq_2 y_2}}$$

Model the ML type constructor *

Given a chain in $D_1 \times D_2$ we need show it has a lub.

$$(x_0, y_0) \sqsubseteq (x_1, y_1) \sqsubseteq \cdots \sqsubseteq (x_n, y_n) \sqsubseteq \cdots$$

Need to define $\bigsqcup_n (x_n, y_n) = (x_\infty, y_\infty)$

$x_\infty \in D_1$, $y_\infty \in D_2$

$$x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots \quad \bigsqcup_n x_n \in D_1$$

$$y_0 \sqsubseteq y_1 \sqsubseteq \cdots \sqsubseteq y_n \sqsubseteq \cdots \quad \bigsqcup_n y_n \in D_2$$

Define $x_\infty = \bigsqcup_n x_n$

$y_\infty = \bigsqcup_n y_n$

Lubs of chains are calculated componentwise:

$$\bigsqcup_{n \geq 0} (d_{1,n}, d_{2,n}) = (\bigsqcup_{i \geq 0} d_{1,i}, \bigsqcup_{j \geq 0} d_{2,j}) \ .$$

If $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ are domains so is $(D_1 \times D_2, \sqsubseteq)$ and $\perp_{D_1 \times D_2} = (\perp_{D_1}, \perp_{D_2})$.

Continuous functions of two arguments

Proposition. *Let D, E, F be cpo's. A function $f : (D \times E) \to F$ is monotone if and only if it is monotone in each argument separately:*

$$\forall d, d' \in D, e \in E.\, d \sqsubseteq d' \;\Rightarrow\; f(d,e) \sqsubseteq f(d',e)$$

$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \;\Rightarrow\; f(d,e) \sqsubseteq f(d,e').$$

Moreover, it is continuous if and only if it preserves lubs of chains in each argument separately:

$$f(\bigsqcup_{m \geq 0} d_m \,,\, e) = \bigsqcup_{m \geq 0} f(d_m, e)$$

$$f(d \,,\, \bigsqcup_{n \geq 0} e_n) = \bigsqcup_{n \geq 0} f(d, e_n).$$

Fn $f : (D \times E) \to F$

is monotone iff$_{\text{def}}$

$\forall (d,e), (d',e') \in D \times E$.

$$\underbrace{(d,e) \sqsubseteq (d',e')}_{\Updownarrow \text{def} \quad \begin{array}{l} d \sqsubseteq d' \text{ in } D \\ e \sqsubseteq e' \text{ in } E \end{array}} \Rightarrow f(d,e) \sqsubseteq f(d',e')$$

- A couple of derived rules:

$$\frac{x \sqsubseteq x' \qquad y \sqsubseteq y'}{f(x,y) \sqsubseteq f(x',y')} \quad (f \text{ monotone})$$

$$\frac{}{f(\bigsqcup_m x_m, \bigsqcup_n y_n) = \bigsqcup_k f(x_k, y_k)} \quad (f \text{ cont})$$

$$\bigsqcup_m f(x_m, \bigsqcup_n y_n) = \bigsqcup_m \bigsqcup_n f(x_m, y_n)$$

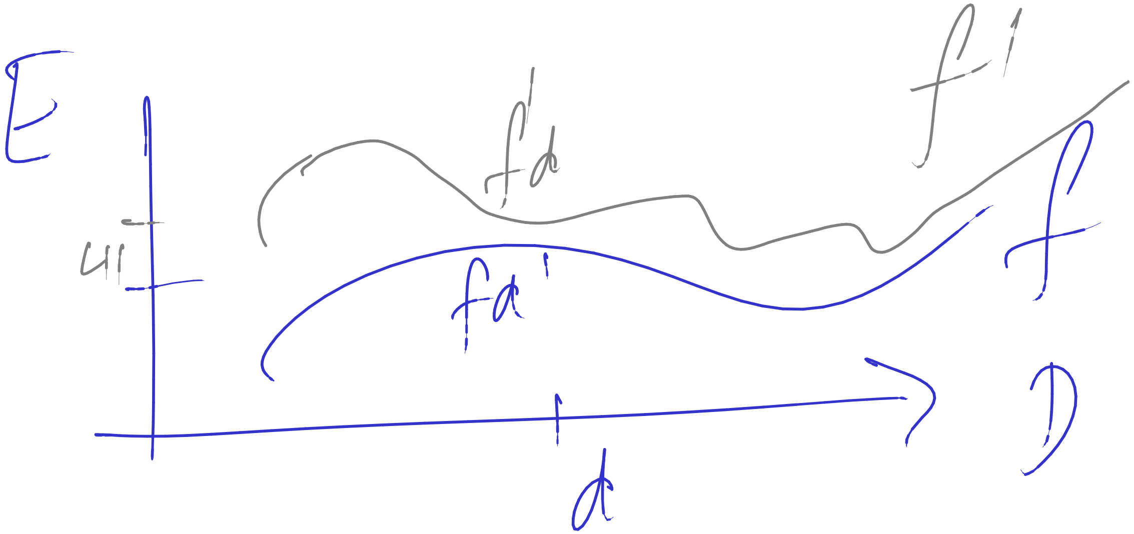
Given cpo's $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$(D \to E) \overset{\text{def}}{=} \{f \mid f : D \to E \text{ is a } \textit{continuous} \text{ function}\}$$

and partial order: $f \sqsubseteq f' \overset{\text{def}}{\Leftrightarrow} \forall d \in D \,.\, f(d) \sqsubseteq_E f'(d)$.

Model the ML type constructor $\to$

Claim: If D and E are domains then so is $(D \to E)$.



Check $(D \to E)$ is a domain whenever D and E are.

- Want a function $\perp_{(D \to E)}$ s.t.

$(D \to E) \ni \perp \sqsubseteq f$ for all cont $f : D \to E$

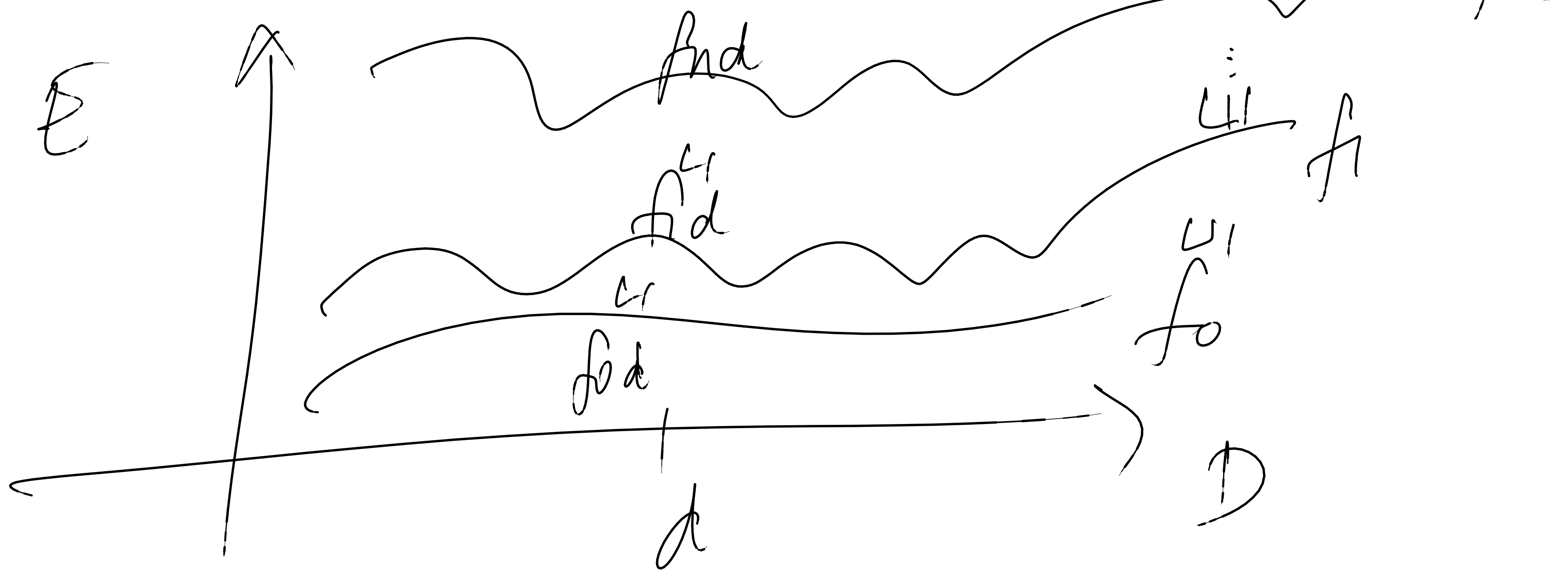
$$\Downarrow$$

$\forall d.\ \perp(d) \sqsubseteq f(d)$

Can be guaranteed by setting $\underset{(D \to E)}{\perp} = \lambda d.\ \perp_E$

◎ Check $(D \to E)$ has lubs of chains

$f_\infty(d) = \bigsqcup_n f_n(d)$ $\qquad f_\infty = \bigsqcup_n f_n$



Def: $f_\infty \overset{def}{=} \lambda d \in D. \bigsqcup_n (f_n(d))$.

f_∞ is continuous and it is a lub of $(f_n)_n$

Want to show f_∞ is continuous.

$$f_\infty\left(\bigsqcup_i d_i\right) \stackrel{?}{=} \bigsqcup_i \left(f_\infty(d_i)\right)$$

by def

$$\bigsqcup_n \left(f_n\left(\bigsqcup_i d_i\right)\right)$$

by cont

$$= \bigsqcup_n \bigsqcup_i \left(f_n(d_i)\right)$$

$$= \bigsqcup_k f_k(d_k)$$

by def

$$\bigsqcup_i \left(\bigsqcup_n \left(f_n(d_i)\right)\right) = \bigsqcup_k f_k(d_k)$$

Function cpo's and domains

Given cpo's $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$(D \to E) \stackrel{\text{def}}{=} \{f \mid f : D \to E \text{ is a } \textit{continuous} \text{ function}\}$$

and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\Leftrightarrow} \forall d \in D \,.\, f(d) \sqsubseteq_E f'(d)$.

- A derived rule:

$$\frac{f \sqsubseteq_{(D \to E)} g \qquad x \sqsubseteq_D y}{f(x) \sqsubseteq g(y)}$$

Lubs of chains are calculated 'argumentwise' (using lubs in E):

$$\bigsqcup_{n\geq 0} f_n \;=\; \lambda d \in D. \bigsqcup_{n\geq 0} f_n(d) \;.$$

- A derived rule:

$$\frac{}{(\bigsqcup_n f_n)(\bigsqcup_m x_m) = \bigsqcup_k f_k(x_k)}$$

If E is a domain, then so is $D \to E$ and $\perp_{D\to E}(d) = \perp_E$, all $d \in D$.

Continuity of composition

For cpo's D, E, F, the composition function

$$\circ : \big((E \to F) \times (D \to E)\big) \longrightarrow (D \to F)$$

defined by setting, for all $f \in (D \to E)$ and $g \in (E \to F)$,

$$g \circ f \;=\; \lambda d \in D.\, g\big(f(d)\big)$$

is continuous.

Continuity of the fixpoint operator

Let D be a domain.

By Tarski's Fixed Point Theorem we know that each continuous function $f \in (D \to D)$ possesses a least fixed point, $fix(f) \in D$.

least pre-fixed point of f

Proposition. *The function*

$$fix : (D \to D) \to D$$

$$f \longmapsto fix(f) \qquad \text{equiv.} \quad \bigsqcup_n f^n(\bot)$$

is continuous.

Show: $fix\left(\bigsqcup_n f_n\right) \stackrel{?}{=} \bigsqcup_n fix(f_n)$

Topic 4

Scott Induction

$\bigsqcup_n f^n \bot = fix(f)$ → Idea: Body of a recursive definition

Scott's Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain D.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of f is in S, *i.e.* that

$$fix(f) \in S \ ,$$

The recursive definition

A property we are interested in

it suffices to prove

$$\forall d \in D \ (d \in S \ \Rightarrow \ f(d) \in S) \ .$$

$\vdots$
$\sqcup \mid$
$f^n \bot$
$\sqcup \mid$
$\vdots$
$\sqcup \mid$
$ff\bot$
$\sqcup \mid$
$f(\bot)$
$\sqcup \mid$
$\bot$

$$\frac{\forall d. \ d \in S \Rightarrow f(d) \in S}{fix(f) \in S}$$

Chain-closed and admissible subsets

Let D be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D

$$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \Big(\bigsqcup_{n \geq 0} d_n\Big) \in S$$

If D is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of D and $\perp \in S$.

Building chain-closed subsets (I)

Let D, E be cpos.

Basic relations:

- For every $d \in D$, the subset

chain closed by definition of lub.

$$\downarrow(d) \stackrel{\text{def}}{=} \{ x \in D \mid x \sqsubseteq d \}$$

of D is chain-closed.

- The subsets

$$\{(x, y) \in D \times D \mid x \sqsubseteq y\}$$

and

$$\{(x, y) \in D \times D \mid x = y\}$$

of $D \times D$ are chain-closed.