NB Suppose x is also a least prefixed point of f, then $x = \mathit{fix}(f)$

Pre-fixed points

Let D be a poset and $f : D \to D$ be a function. Show (1) $\mathit{fix}(f) \sqsubseteq x$

An element $d \in D$ is a pre-fixed point of f if it satisfies $f(d) \sqsubseteq d$. (2) $x \sqsubseteq \mathit{fix}(f)$

$\Rightarrow \mathit{fix}(f) = x$

The *least pre-fixed point* of f, if it exists, will be written

$$\boxed{\mathit{fix}(f)}$$

$\mathit{fix}(f)$ is a pre-fixed point.

It is thus (uniquely) specified by the two properties:

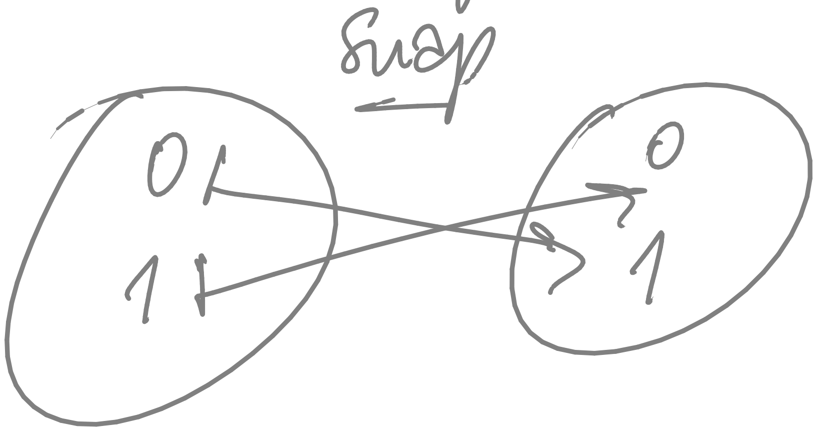
$$f(\mathit{fix}(f)) \sqsubseteq \mathit{fix}(f) \qquad \text{(lfp1)}$$

$$\forall d \in D.\ f(d) \sqsubseteq d \;\Rightarrow\; \mathit{fix}(f) \sqsubseteq d. \qquad \text{(lfp2)}$$

it is least amongst all pre-fixed points.

$f : D \to D$ monotone for D a poset

? do such function always have fix ?



$x \sqsubseteq y \Rightarrow fx \sqsubseteq fy$

Proof principle

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\mathit{fix}(f) \in D$.

   For all $x \in D$, to prove that $\mathit{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

Proof principle

Example
$$\frac{f_{[\![B]\!],[\![C]\!]}([\![P]\!]) \sqsubseteq [\![P]\!]}{fix(f_{[\![B]\!],[\![C]\!]}) = [\![\text{while } B \text{ do } C]\!] \sqsubseteq [\![P]\!]}$$

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $fix(f) \in D$.

   For all $x \in D$, to prove that $fix(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

$$\frac{f(x) \sqsubseteq x}{fix(f) \sqsubseteq x}$$

Proof principle

1.

$$\frac{}{f(\mathit{fix}(f)) \sqsubseteq \mathit{fix}(f)}$$

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\mathit{fix}(f) \in D$.

   For all $x \in D$, to prove that $\mathit{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

$$\frac{f(x) \sqsubseteq x}{\mathit{fix}(f) \sqsubseteq x}$$

If $\mathrm{fix}(f)$ exists, then $f(\mathrm{fix}(f)) = \mathrm{fix}(f)$.

$$\frac{x \sqsubseteq y}{fx \sqsubseteq fy}$$

$$\frac{}{f(\mathrm{fix}\, f) \sqsubseteq \mathrm{fix}\, f}\checkmark$$

Least pre-fixed points are fixed points

If it exists, the least pre-fixed point of a mononote function on a partial order is necessarily a fixed point.

$$\frac{\dfrac{\dfrac{}{f(\mathrm{fix}\, f) \sqsubseteq \mathrm{fix}\, f}\checkmark \qquad \dfrac{f(f(\mathrm{fix}\, f)) \sqsubseteq f(\mathrm{fix}\, f)}{\mathrm{fix}\, f \sqsubseteq f(\mathrm{fix}\, f)}}{}}{f(\mathrm{fix}\, f) = \mathrm{fix}(f)}$$

gives a notion of passage to the limit.

Eg. $D = (State \to State)$

Thesis*

$f_0 \subseteq f_1 \subseteq \cdots \subseteq f_n \subseteq \cdots f_\infty$

$\quad = \bigcup_n f_n$

> All domains of computation are complete partial orders with a least element.

Thesis*

All domains of computation are
complete partial orders with a least element.

All computable functions are
continuous.

$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \quad d_\infty$

$e_0 = f(d_0) \sqsubseteq e_1 = f(d_1) \sqsubseteq e_2 = f(d_2) \sqsubseteq \cdots \sqsubseteq e_n = f d_n \sqsubseteq \cdots$

continuity

$e_\infty = f(d_\infty)$

$D \xrightarrow{f} E$

lub of $(d_n)_n$

$d \sqsupseteq \bigsqcup d_n$

A chain complete poset, or cpo for short, is a poset $(D, \sqsubseteq)$ in which all countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ have least upper bounds, $\bigsqcup_{n\geq 0} d_n$:

$$\forall m \geq 0 \,.\, d_m \sqsubseteq \bigsqcup_{n\geq 0} d_n \qquad \text{(lub1)}$$

$$\forall d \in D \,.\, (\forall m \geq 0 \,.\, d_m \sqsubseteq d) \;\Rightarrow\; \bigsqcup_{n\geq 0} d_n \sqsubseteq d. \qquad \text{(lub2)}$$

$\sqcup\!|$

$\vdots$

$\sqcup\!|$

d_n

$\vdots$

$\sqcup\!|$

d_1

$\sqcup\!|$

d_0

A domain is a cpo that possesses a least element, $\bot$:

$$\forall d \in D \,.\, \bot \sqsubseteq d.$$

cf. $\dfrac{f(x) \sqsubseteq x}{\underline{fix}(f) \sqsubseteq x}$

$$\frac{}{\bot \sqsubseteq x}$$

$$\frac{}{x_i \sqsubseteq \bigsqcup_{n \geq 0} x_n} \quad (i \geq 0 \text{ and } \langle x_n \rangle \text{ a chain})$$

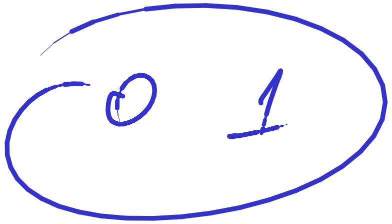
make sure the x_n form a chain

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq x}{\bigsqcup_{n \geq 0} x_n \sqsubseteq x} \quad (\langle x_i \rangle \text{ a chain})$$

Domain of partial functions, $X \rightharpoonup Y$

Remark. Not every poset is a domain.

E.g. it could lack a least element



This can however be resolved by adding a new least element.

$0 \sqsubseteq 0,\ 1 \sqsubseteq 1,\ \bot \sqsubseteq \bot$

$\bot \sqsubseteq 0,\ \bot \sqsubseteq 1$

But there are other possibilities for counterexamples

$0 \sqsubseteq 1 \sqsubseteq 2 \sqsubseteq \cdots \sqsubseteq n \sqsubseteq \cdots$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$f \sqsubseteq g \quad \text{iff} \quad dom(f) \subseteq dom(g) \text{ and } \forall x \in dom(f).\ f(x) = g(x)$$

$$\text{iff} \quad graph(f) \subseteq graph(g)$$

$x \in dom(f_n)$

Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

$\exists n \in \mathbb{N}.$

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

$x \in \bigcup_n dom(f_n)$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$
\begin{array}{lll}
f \sqsubseteq g & \text{iff} & dom(f) \subseteq dom(g) \text{ and} \\
 & & \forall x \in dom(f).\ f(x) = g(x) \\
 & \text{iff} & graph(f) \subseteq graph(g)
\end{array}
$$

Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

$$
f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n)\text{, some } n \\ \text{undefined} & \text{otherwise} \end{cases}
$$

Least element $\perp$ is the totally undefined partial function.

$d \sqsubseteq d \sqsubseteq d \sqsubseteq \cdots \sqsubseteq d \sqsubseteq \cdots$ $\bigsqcup_n d = d$

$\forall n.\ d_n \sqsubseteq \bigsqcup_n d_{N+n}$

Some properties of lubs of chains

$\bigsqcup_n d_n \sqsubseteq \bigsqcup_n d_{N+n}$

Let D be a cpo.

1. For $d \in D$, $\bigsqcup_n d = d$.

$\bigsqcup d_{N+n} \sqsubseteq \bigsqcup d_n$

2. For every chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ in D,

$$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$$

for all $N \in \mathbb{N}$.

$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_n \cdots \bigsqcup_n d_n$

$\parallel$

$d_N \sqsubseteq d_{N+1} \sqsubseteq \cdots \sqsubseteq d_{N+k} \cdots \bigsqcup_k d_{N+k}$

claim

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \ \sqsubseteq \ \bigsqcup_n e_n$.

$$\begin{array}{ccccccccc}
e_0 & \sqsubseteq & e_1 & \sqsubseteq \cdots \sqsubseteq & e_n & \sqsubseteq \cdots & \bigsqcup e_n \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & \sqsupseteq \\
d_0 & \sqsubseteq & d_1 & \sqsubseteq \cdots \sqsubseteq & d_n & \sqsubseteq \cdots & \bigsqcup d_n
\end{array}$$

$$\dfrac{\dfrac{\forall n \qquad \dfrac{\checkmark}{d_n \sqsubseteq e_n} \qquad \dfrac{\checkmark}{e_n \sqsubseteq \bigsqcup_n e_n}\,\text{lub}}{\forall n \qquad d_n \sqsubseteq \bigsqcup_n e_n}}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n}\,\text{lub2}$$

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

   if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \ \sqsubseteq \ \bigsqcup_n e_n$.

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq y_n}{\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n} \quad (\langle x_n \rangle \text{ and } \langle y_n \rangle \text{ chains})$$