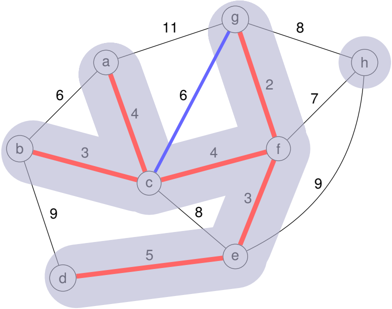



# 6.3: Minimum Spanning Tree

Frank Stajano Thomas Sauerwald

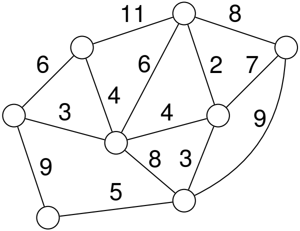
Lent 2015

UNIVERSITY OF CAMBRIDGE

# Minimum Spanning Tree Problem

**Minimum Spanning Tree Problem**

- Given: undirected, connected graph $G = (V, E, w)$ with non-negative edge weights



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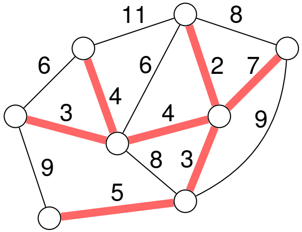

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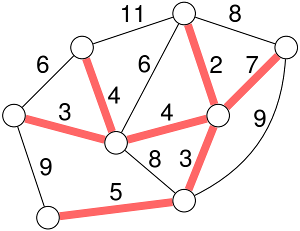
Must be necessarily a tree!



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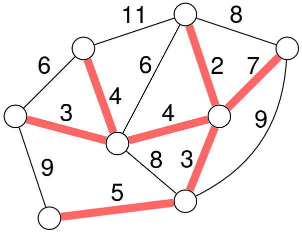


**Applications**

- Street Networks, Wiring Electronic Components, Laying Pipes
- Weights may represent distances, costs, travel times, capacities, resistance etc.

# Generic Algorithm

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def minimum spanningTree(G)
  A = empty set of edges
  while A does not span all vertices yet:
    add a safe edge to A
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**How to find a safe edge?**

# Finding safe edges

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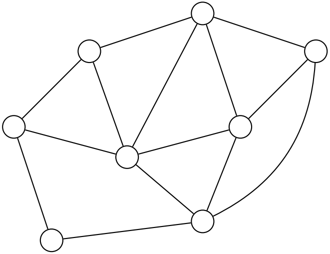

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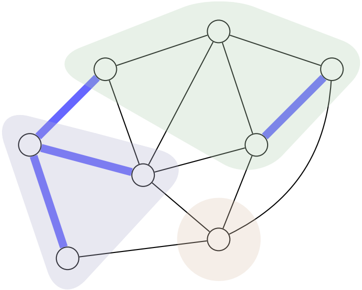

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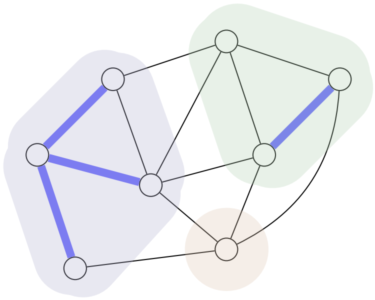

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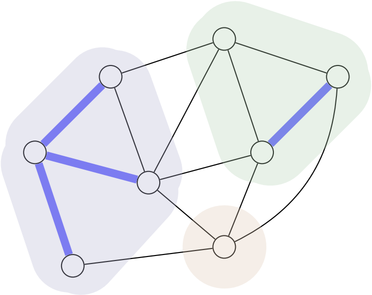

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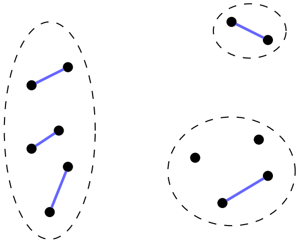

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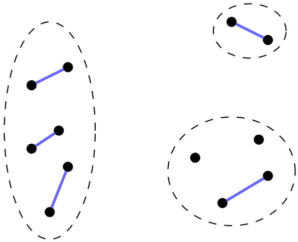

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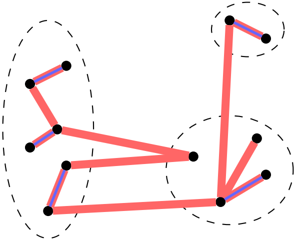

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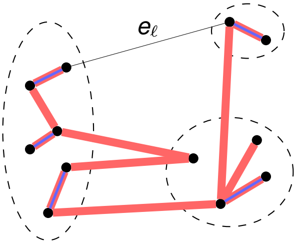

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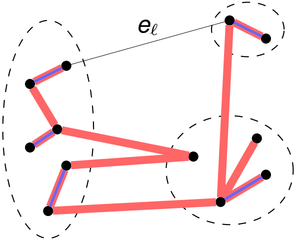

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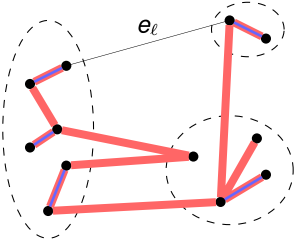

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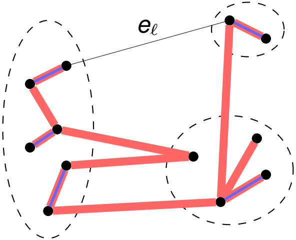

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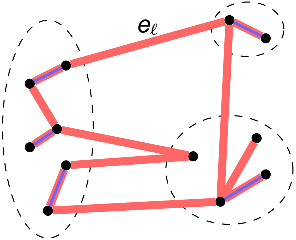

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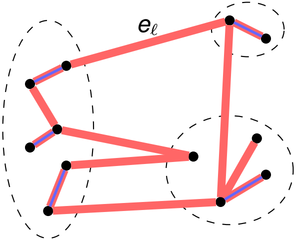

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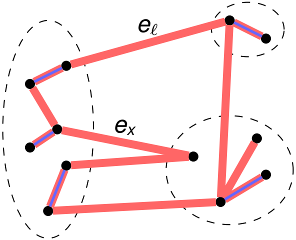

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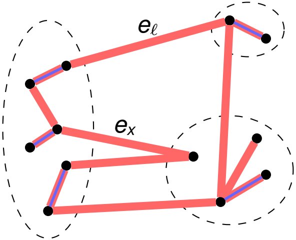

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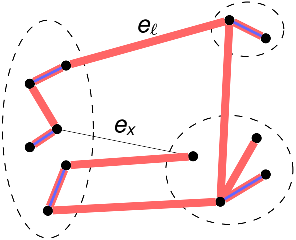

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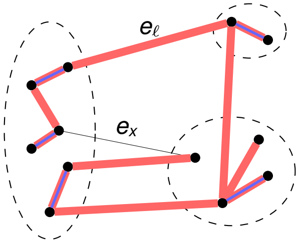

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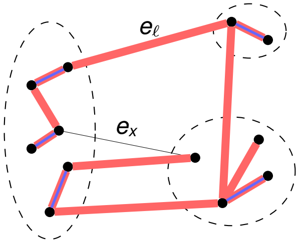

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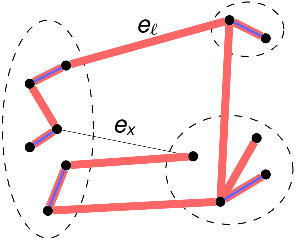
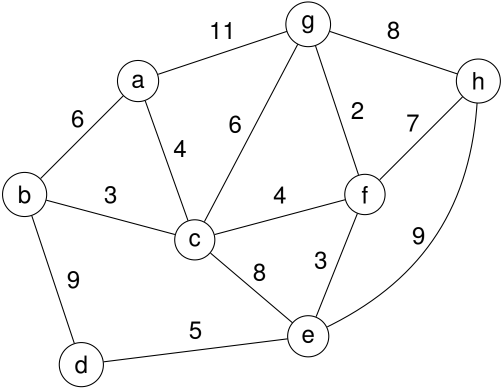


## Glimpse at Kruskal's Algorithm

Basic Strategy



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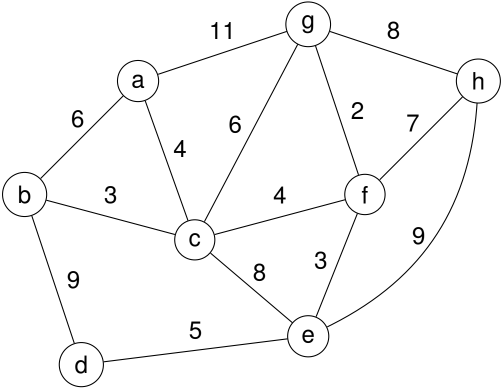

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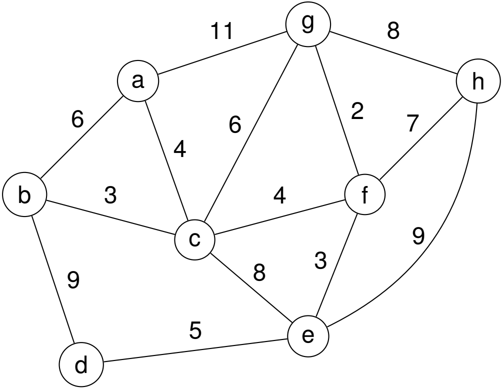

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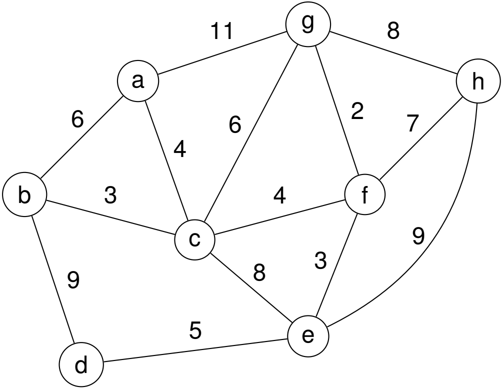

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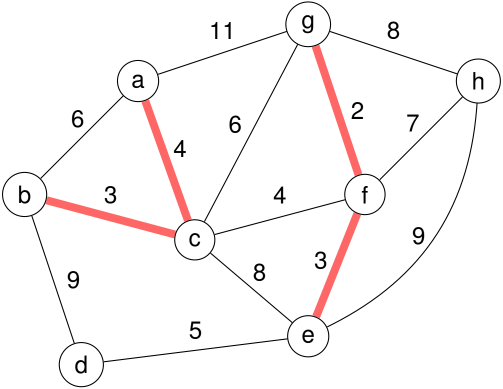

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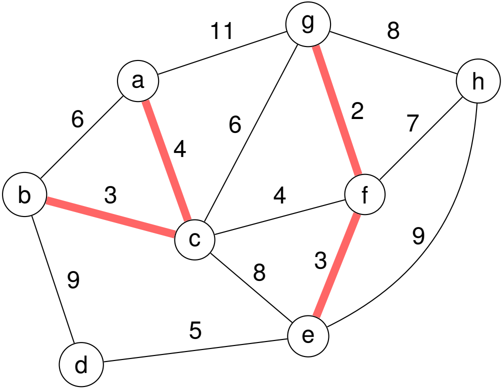

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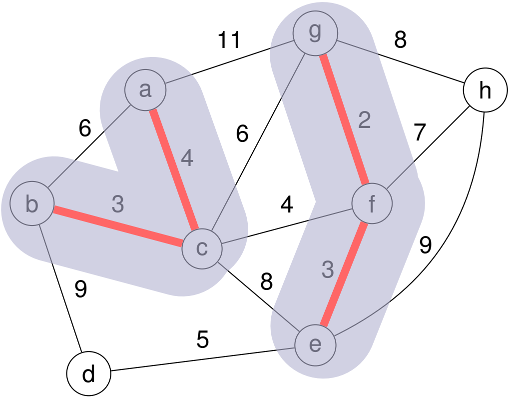

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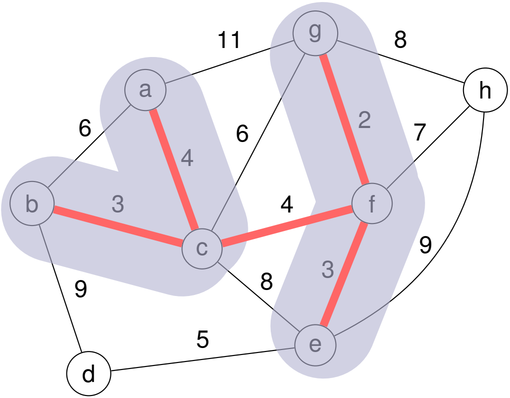

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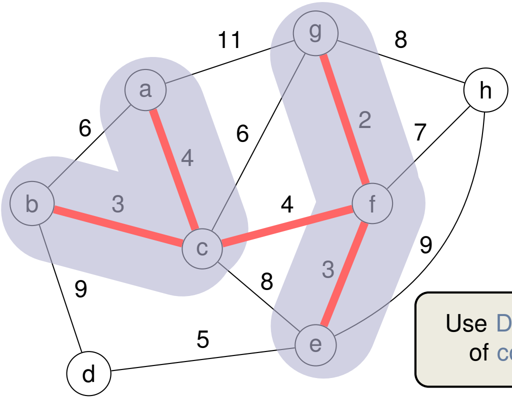


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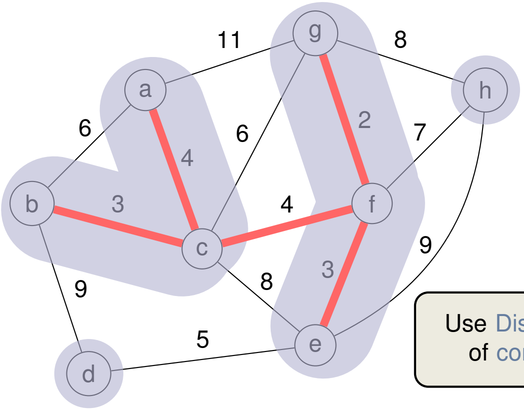


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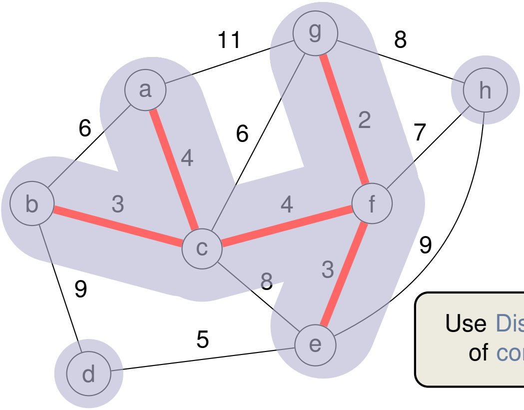


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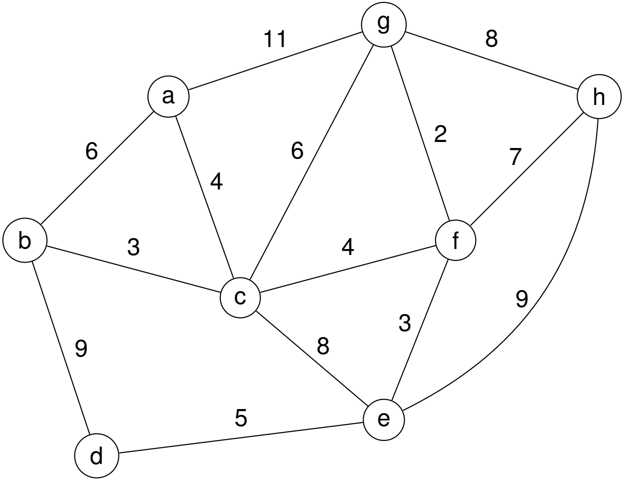

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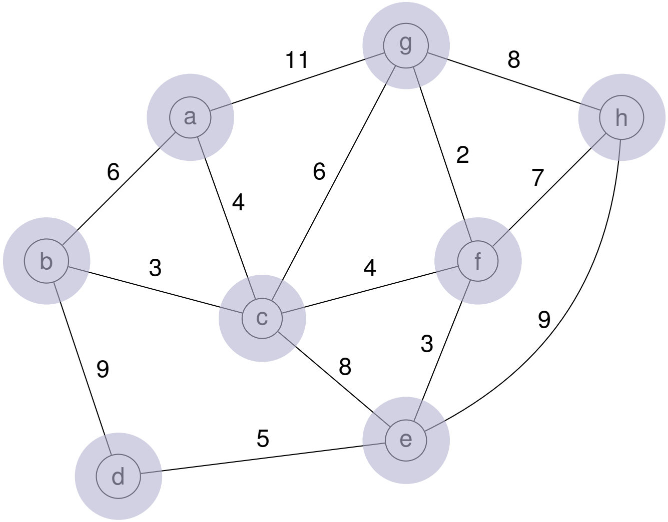

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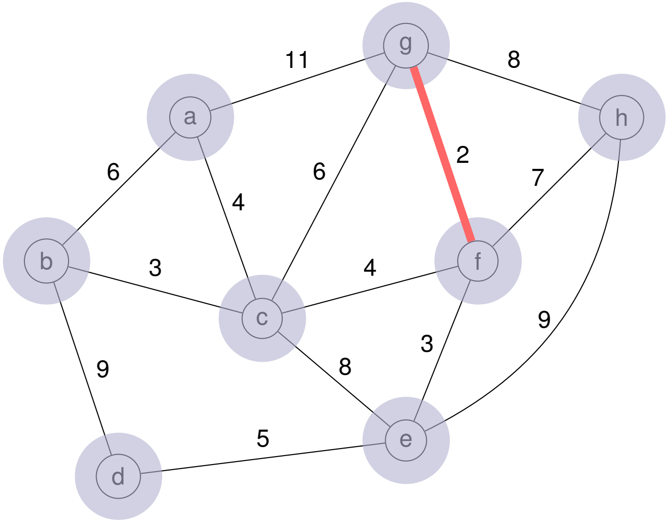

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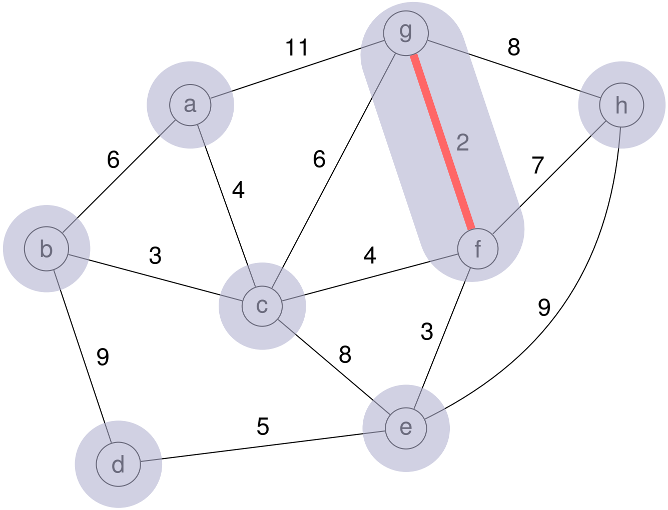

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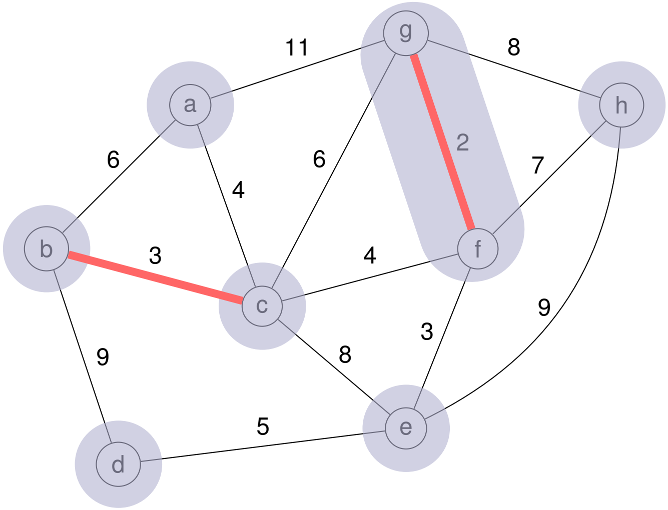

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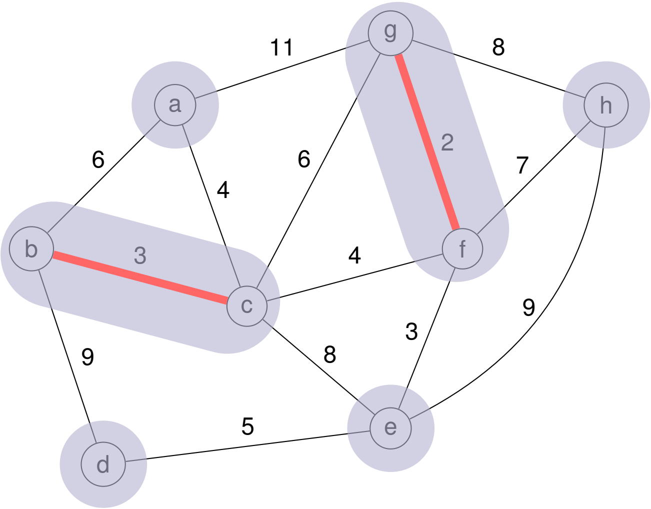

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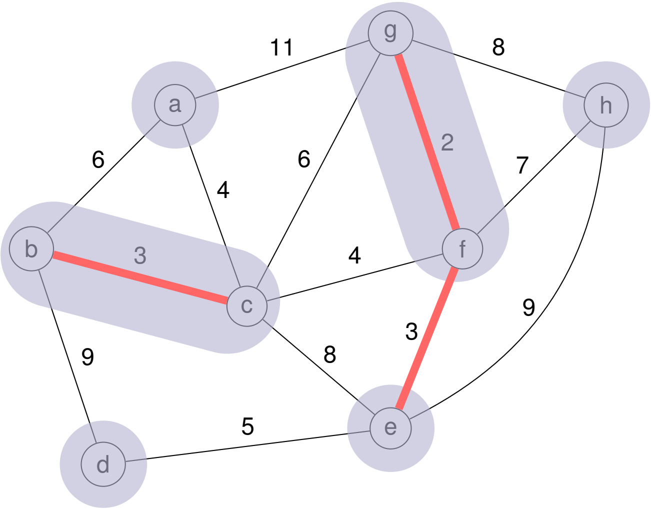

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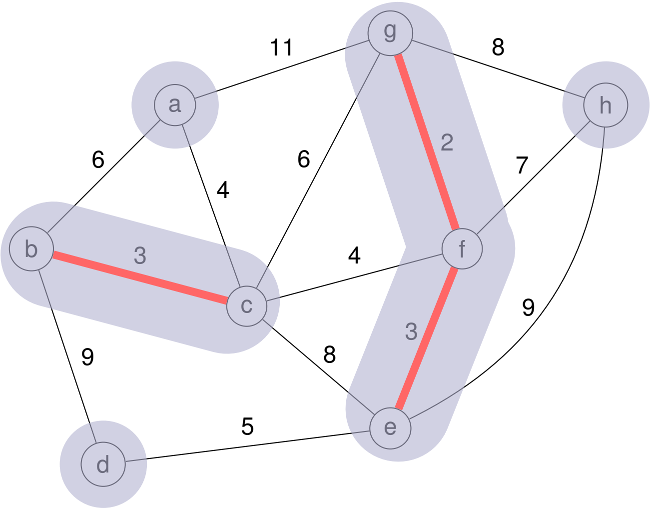

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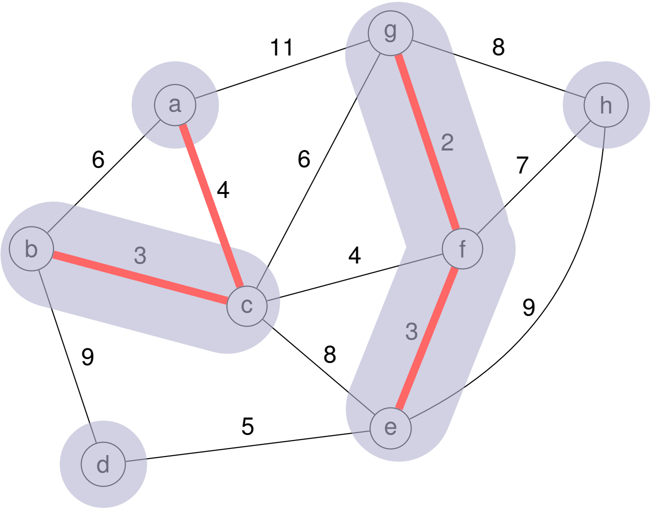

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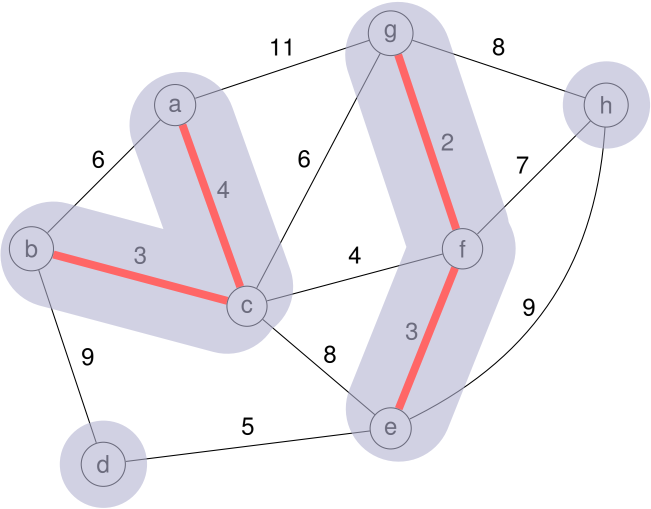

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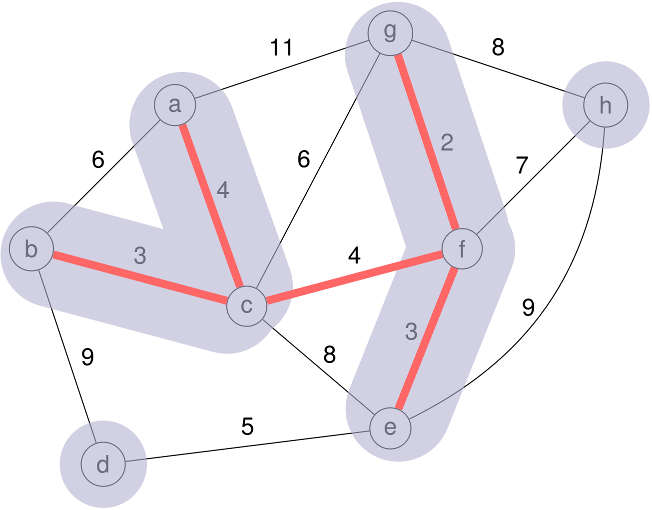

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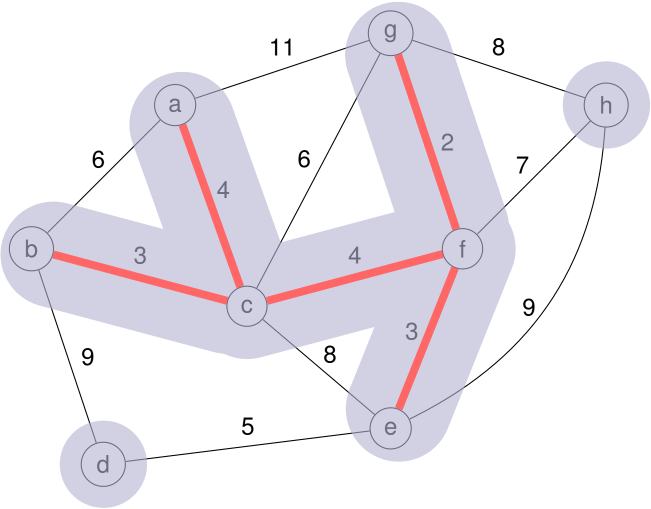

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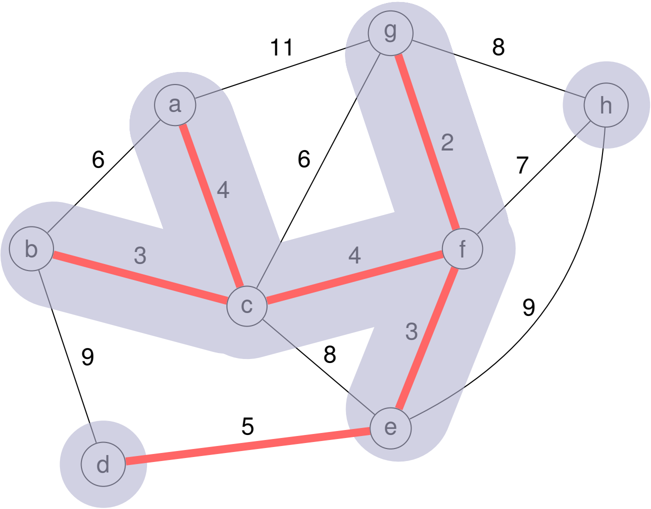

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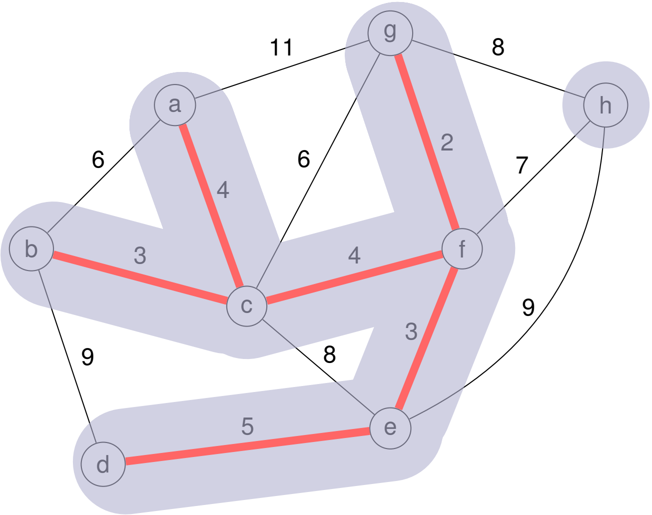

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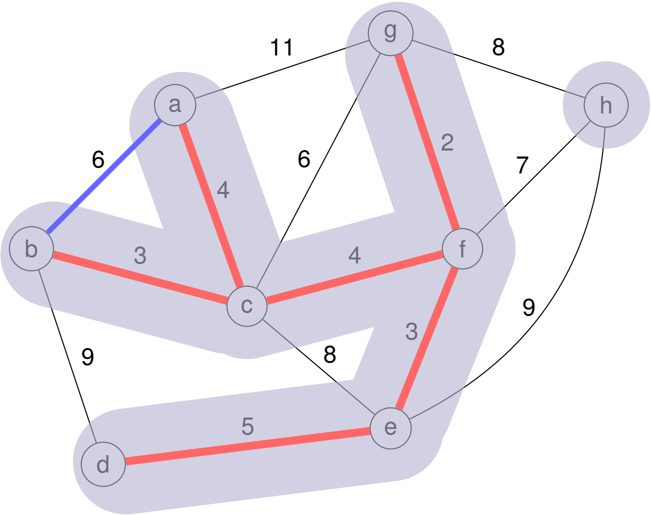

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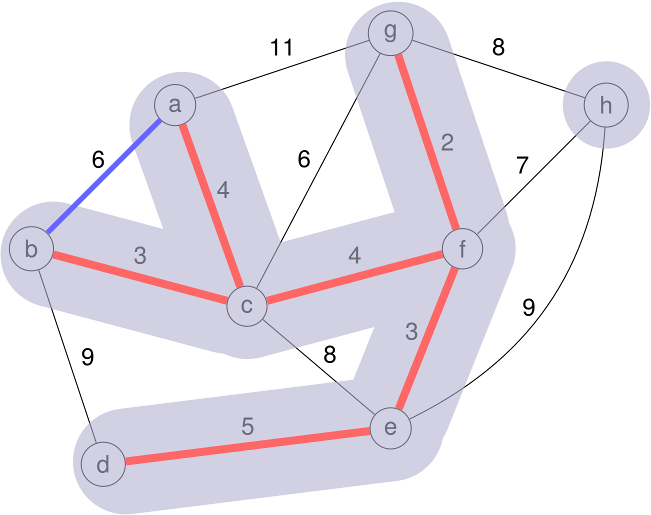

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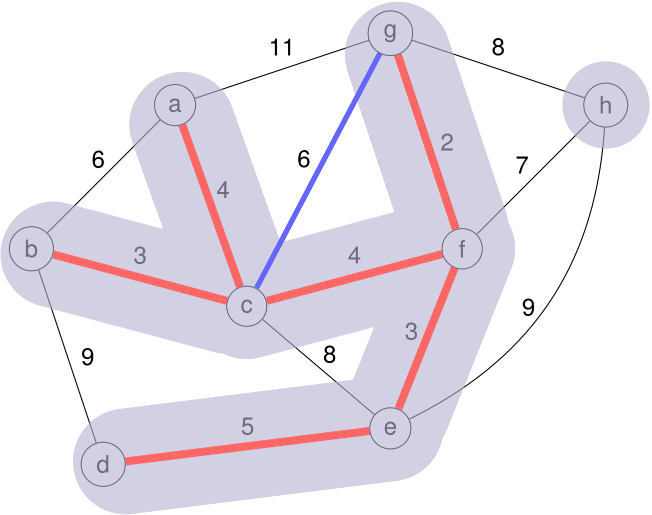

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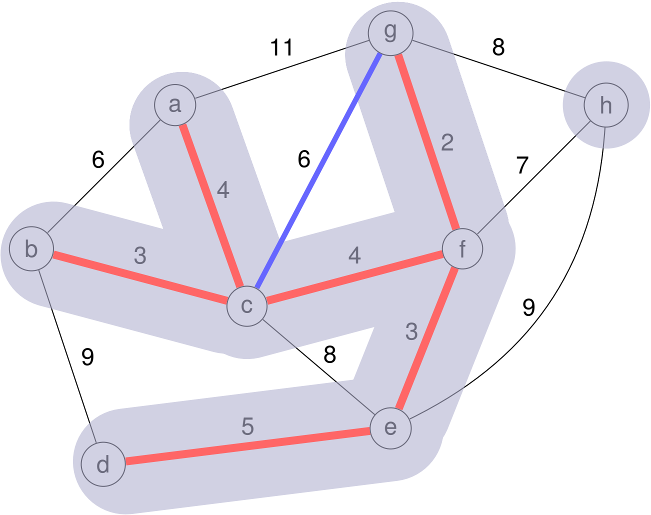

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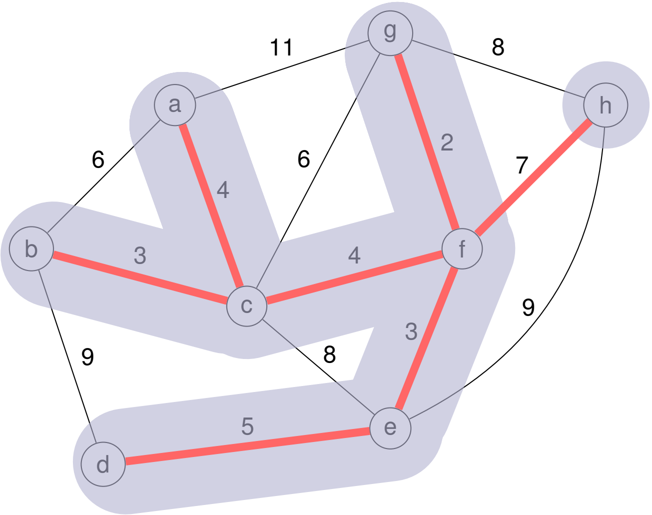

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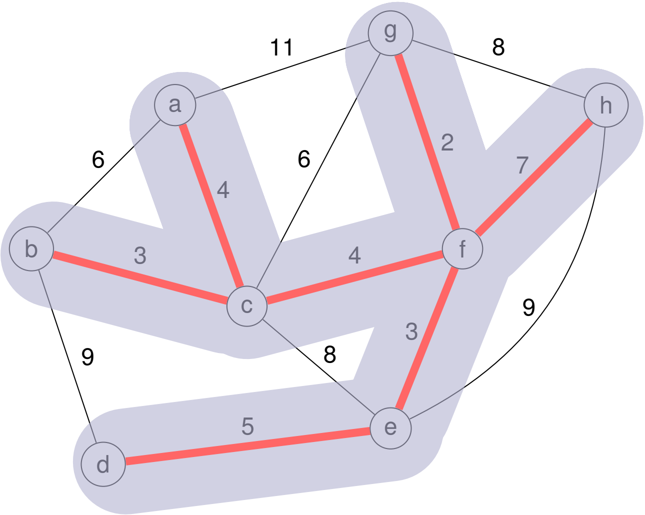

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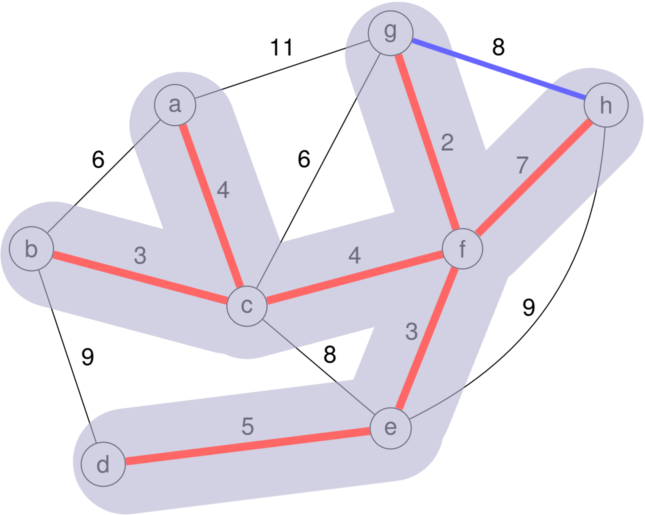

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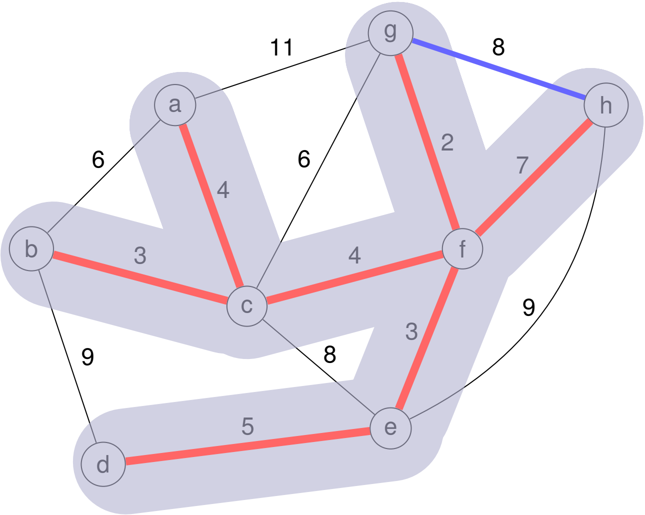

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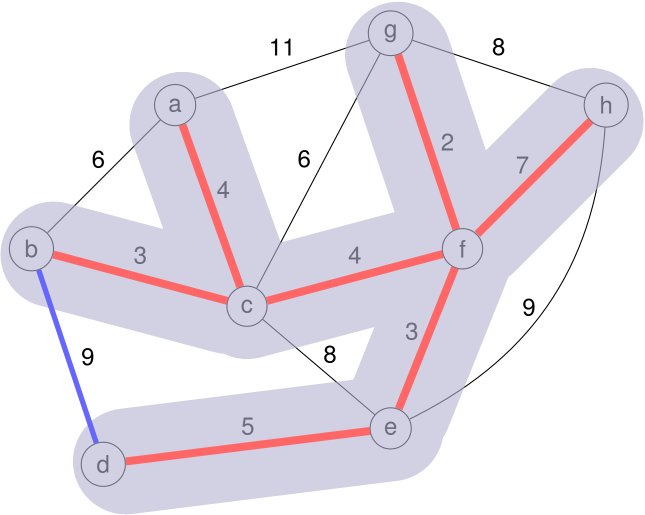

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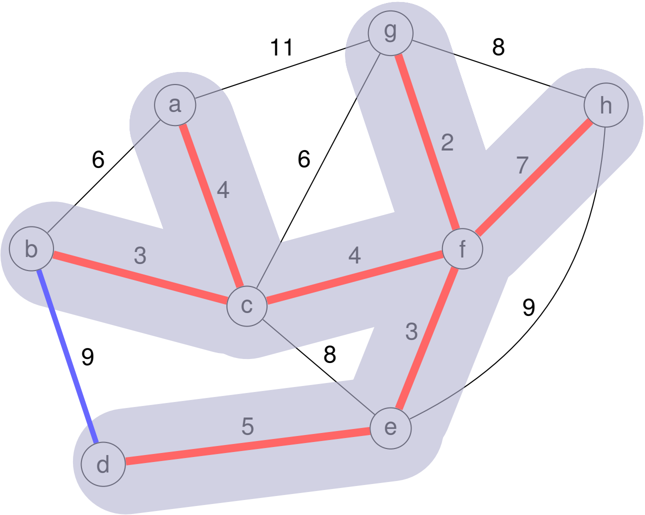

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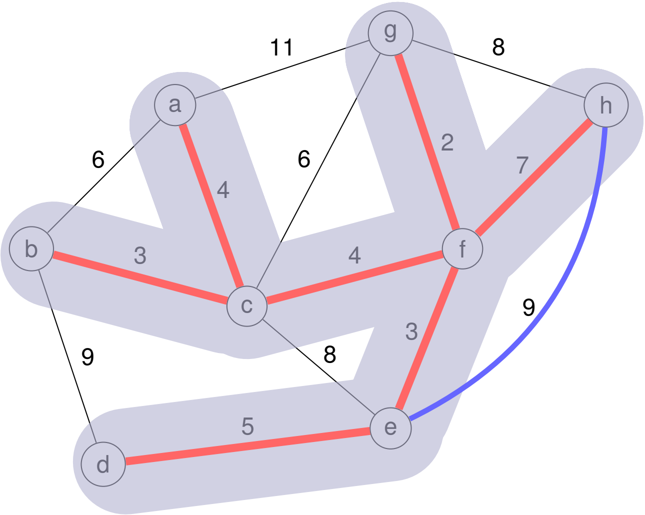

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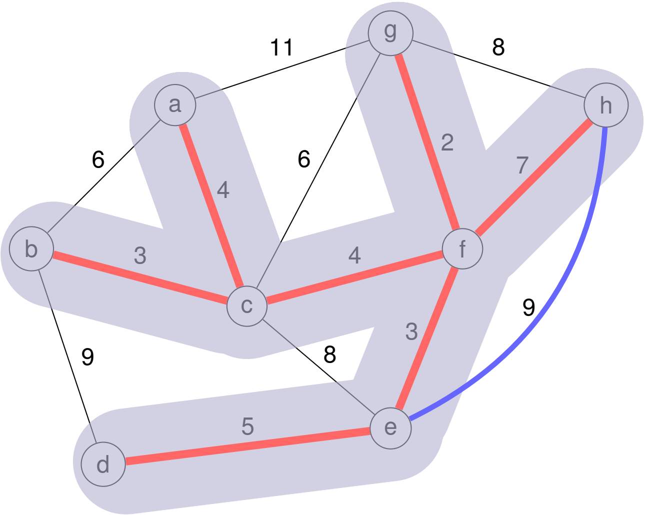

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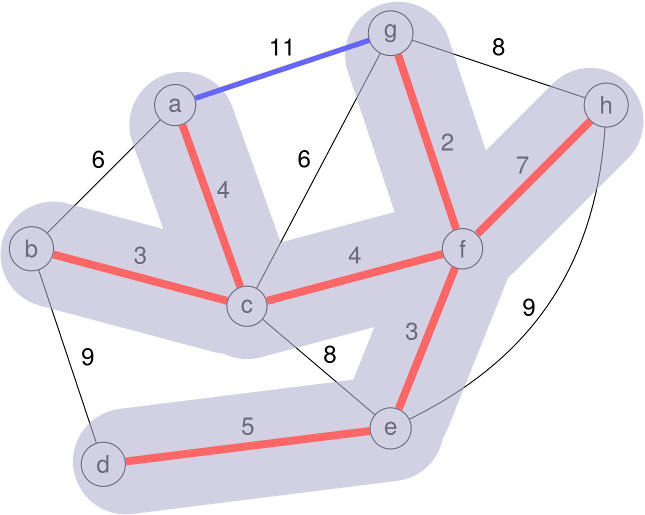

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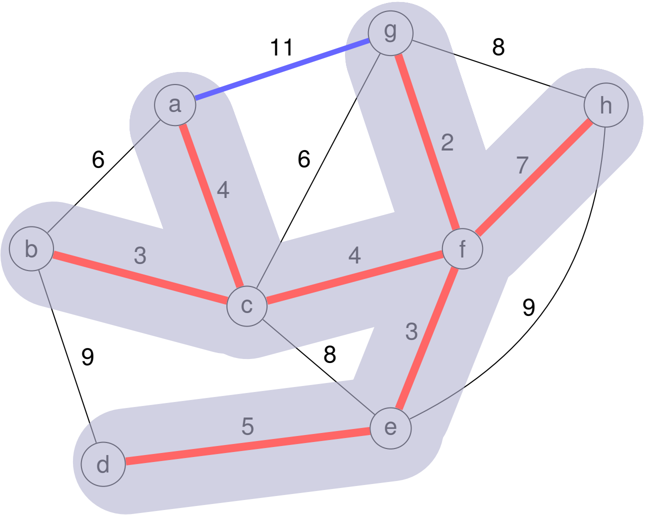

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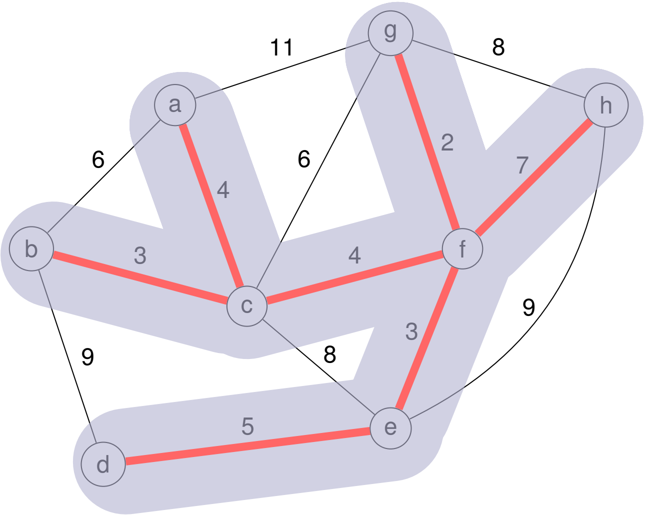

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If edges are already sorted, runtime becomes $O(E \cdot \alpha(n))$!

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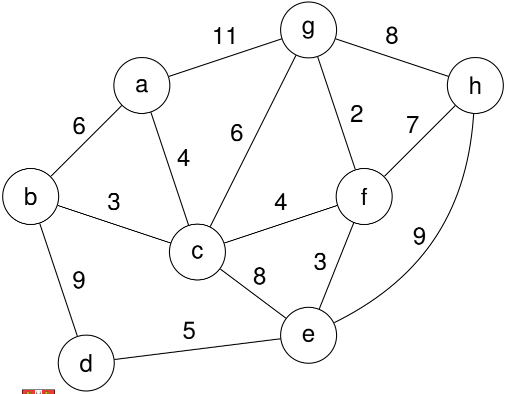
**Correctness**

- Consider the cut of all connected components (disjoint sets)
- L. 14 ensures that we extend $A$ by an edge that goes across the cut
- This edge is also the lightest edge crossing the cut (otherwise, we would have included a lighter edge before)

# Prim's Algorithm

Basic Strategy

- Start growing a tree from a designated root vertex



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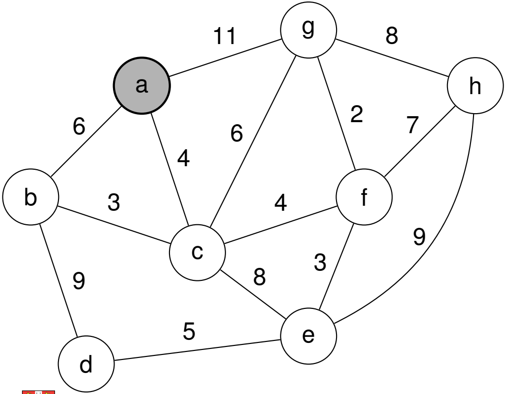

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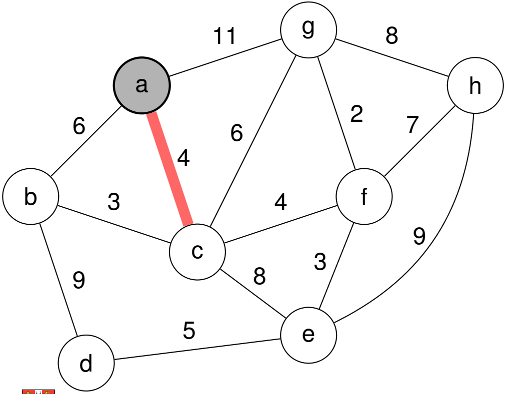

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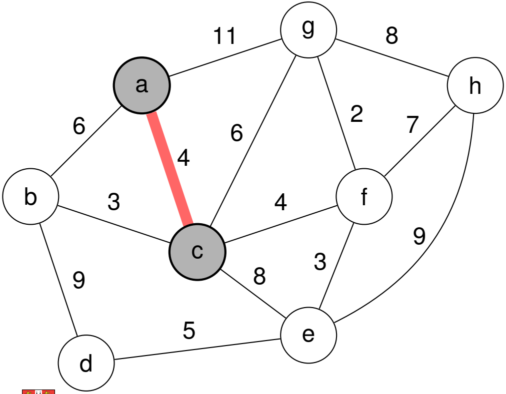

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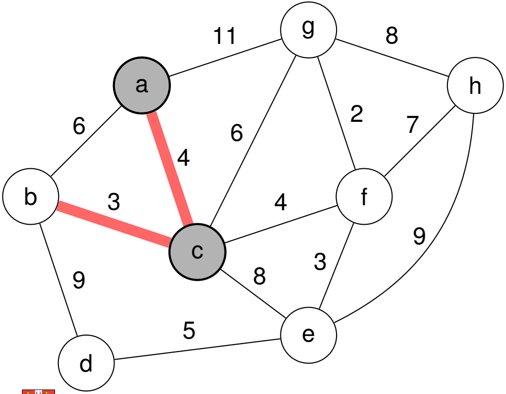

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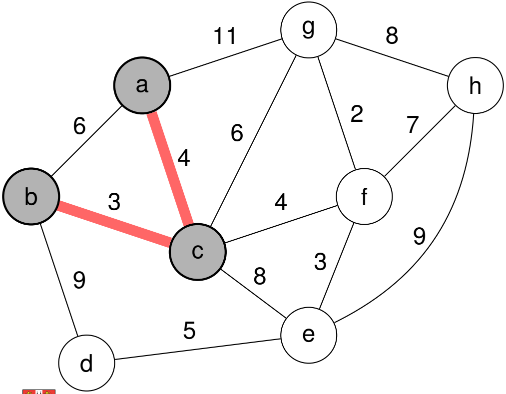

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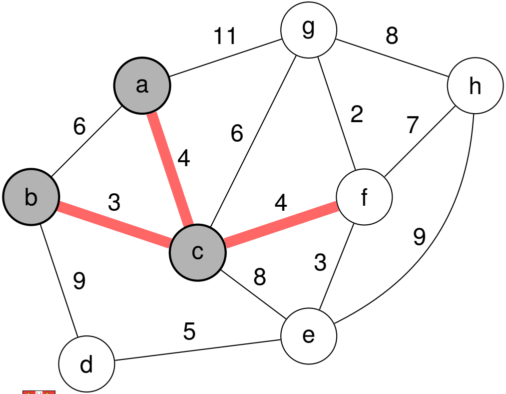

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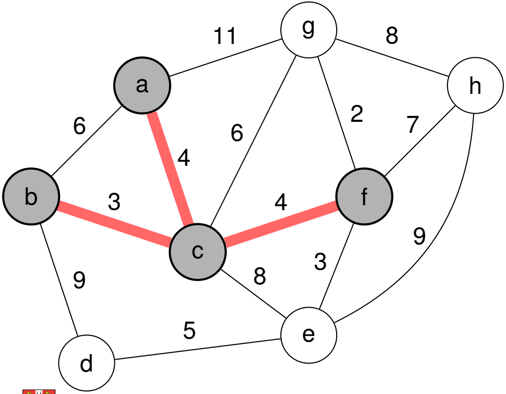

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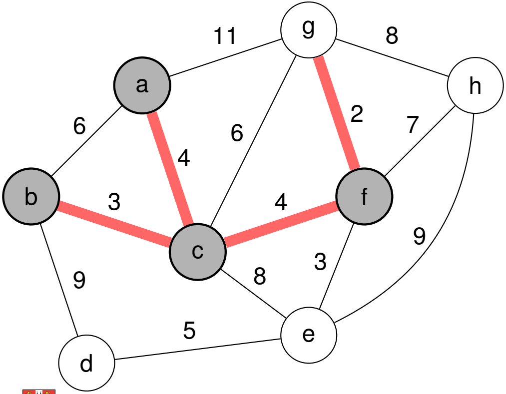

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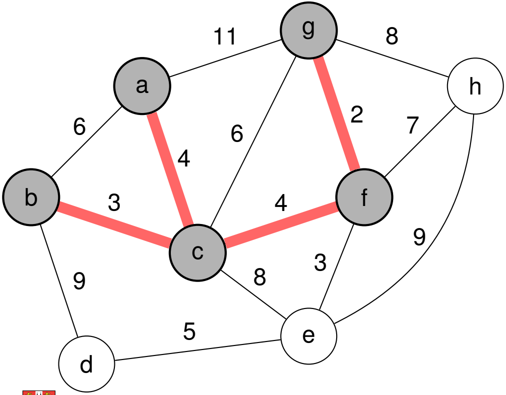

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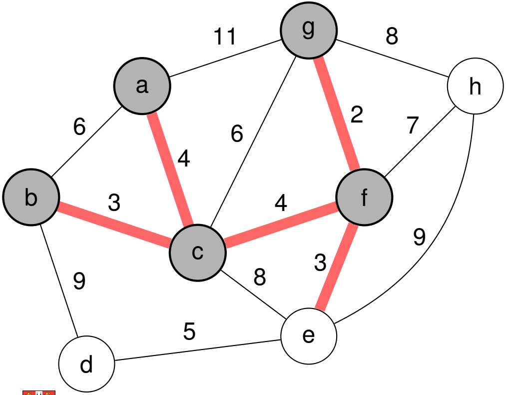

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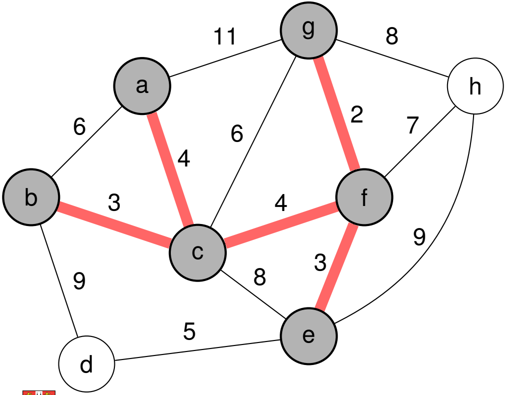

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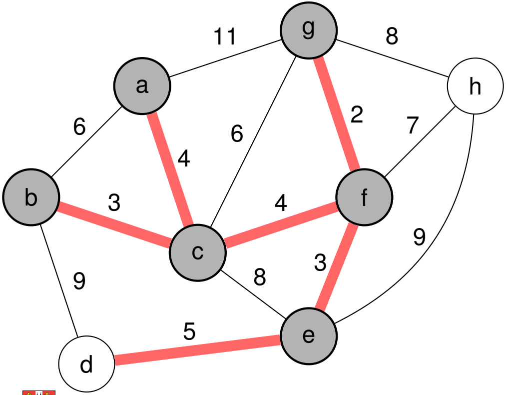

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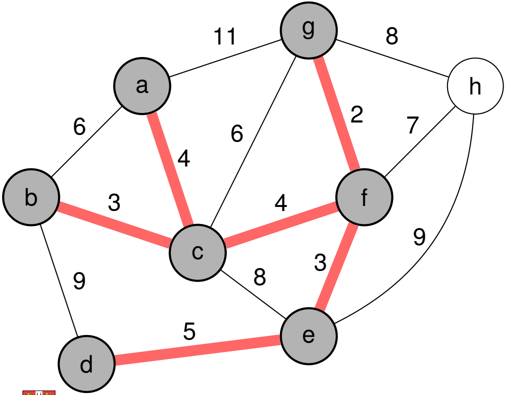

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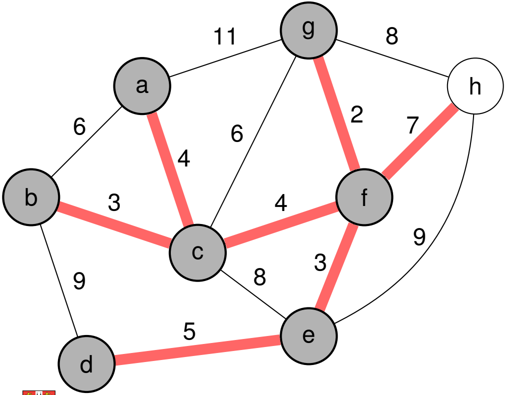

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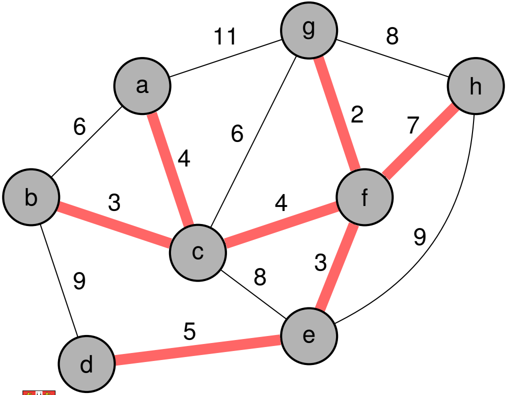

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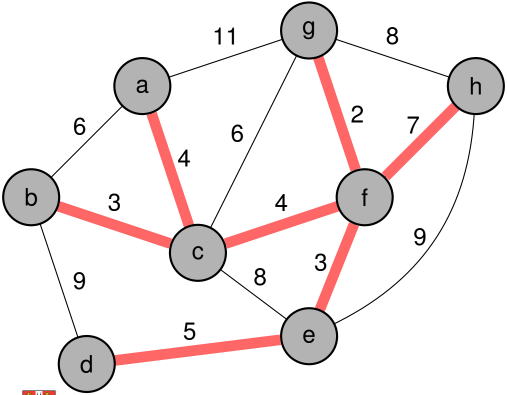
Implementation will be based on vertices!



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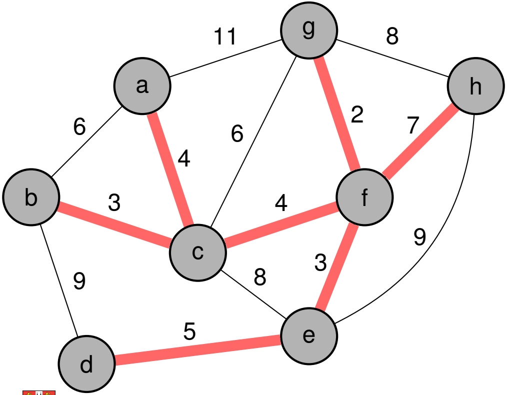
Assign every vertex not in $A$ a key which is at all stages equal to the smallest weight of an edge connecting to $A$



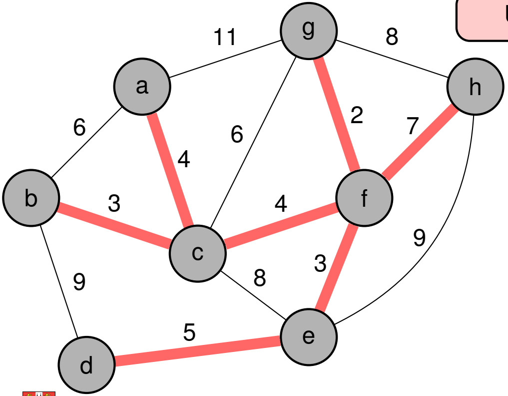
# Prim's Algorithm

Basic Strategy

- Start growing a tree from a designated root vertex
- At each step, add lightest edge linked to $A$ that does not yield cycle

Assign every vertex not in $A$ a key which is at all stages equal to the smallest weight of an edge connecting to $A$

Use a Priority Queue!



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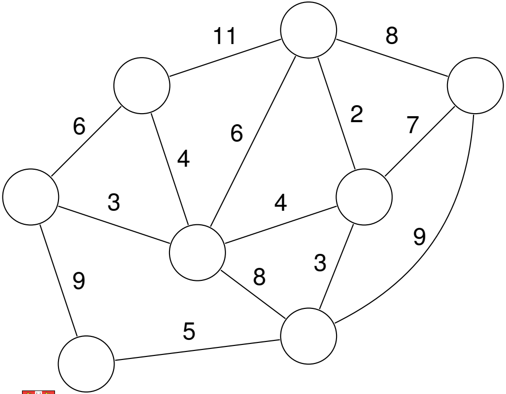

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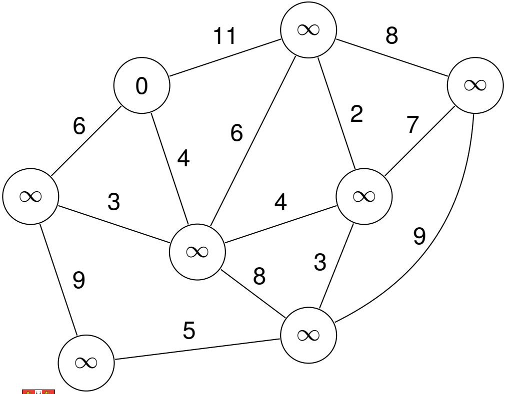

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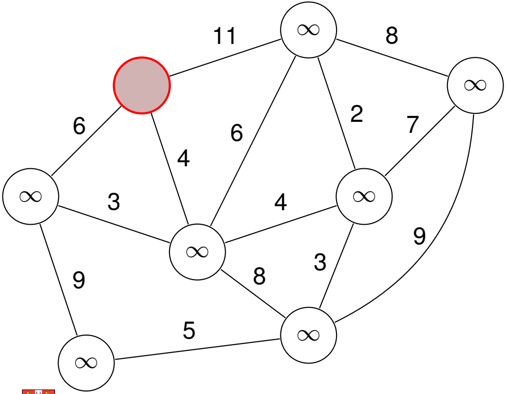

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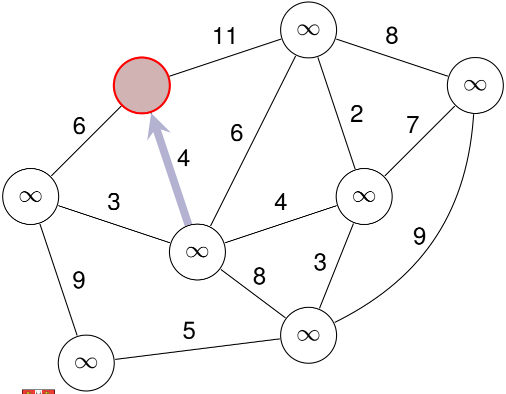

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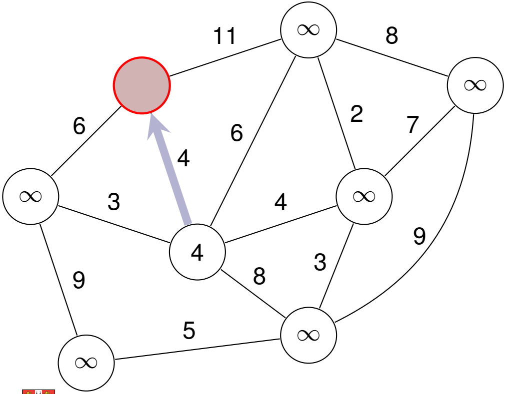

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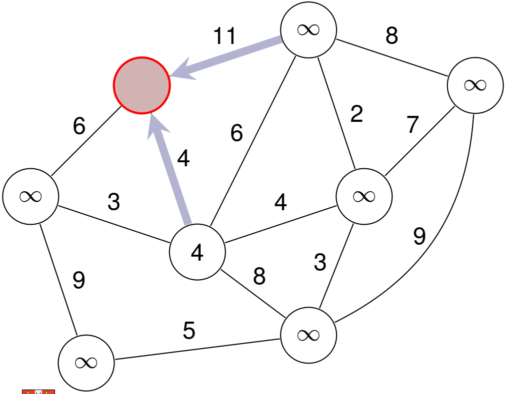

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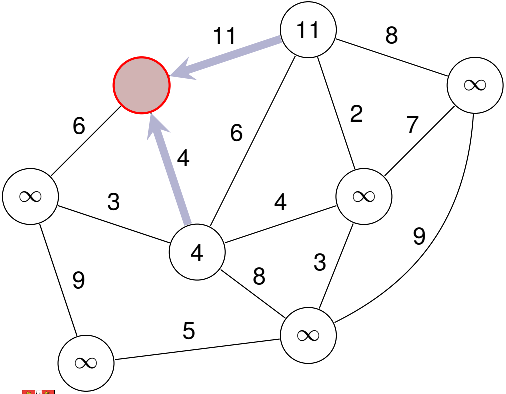

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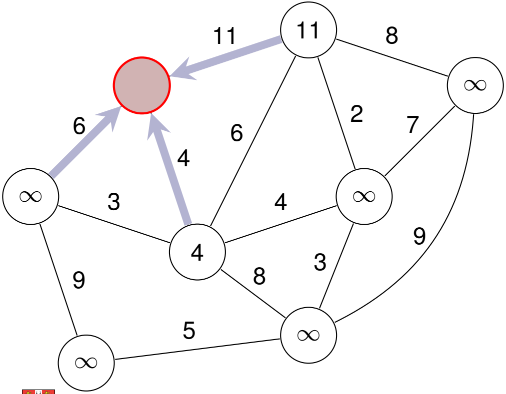

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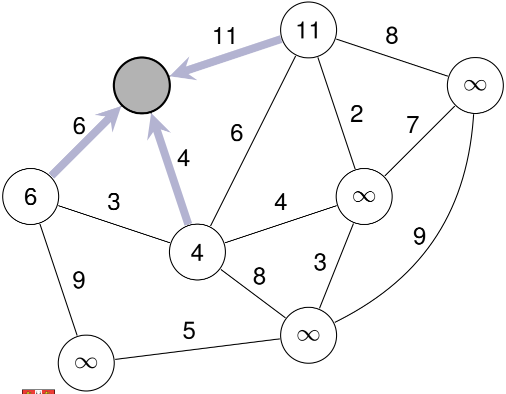

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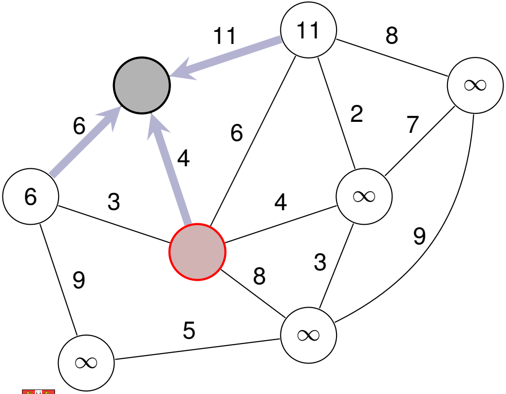

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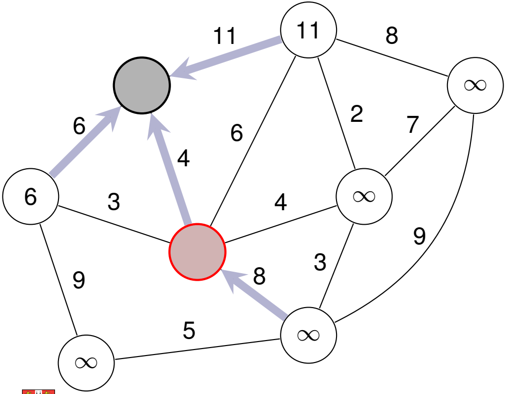

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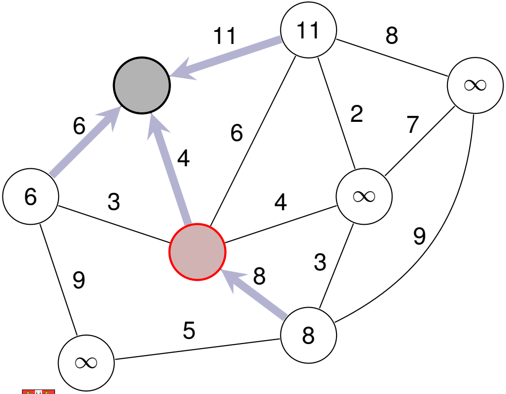

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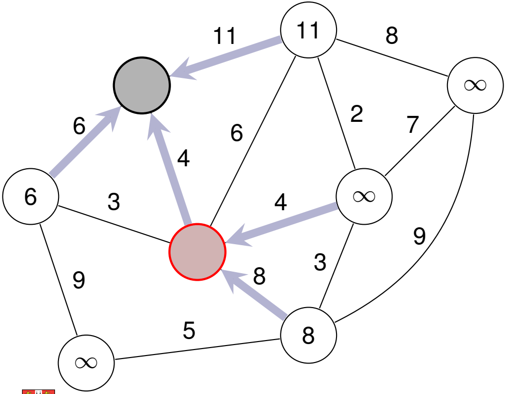

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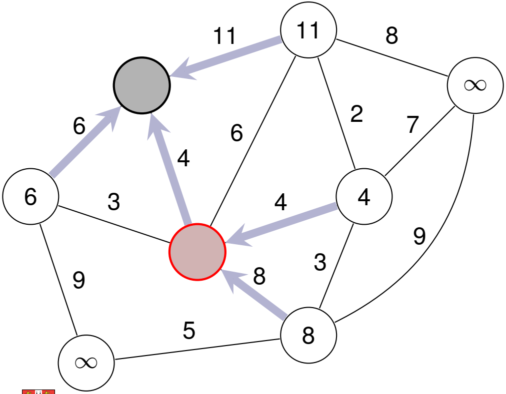

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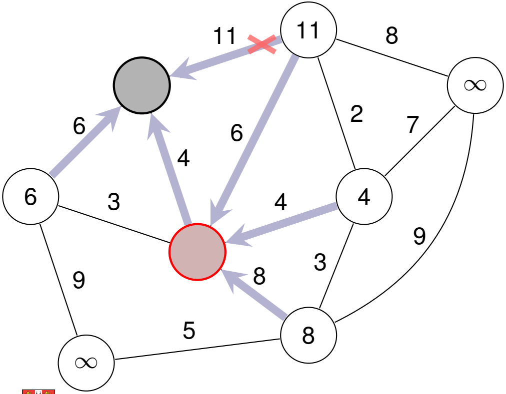

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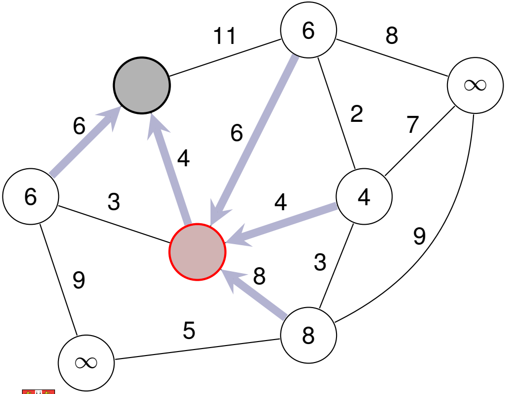

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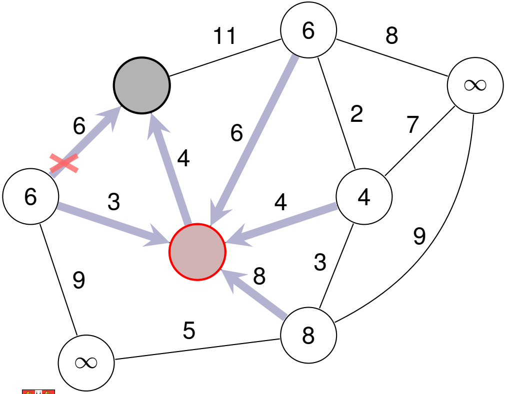

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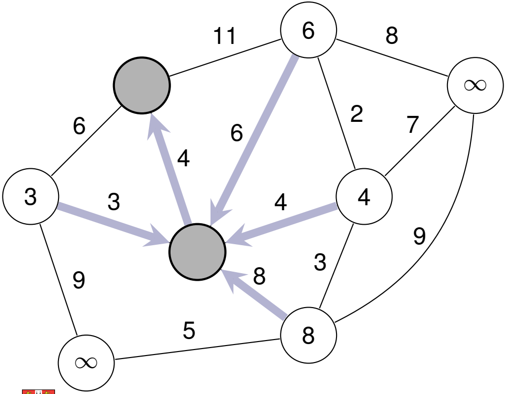

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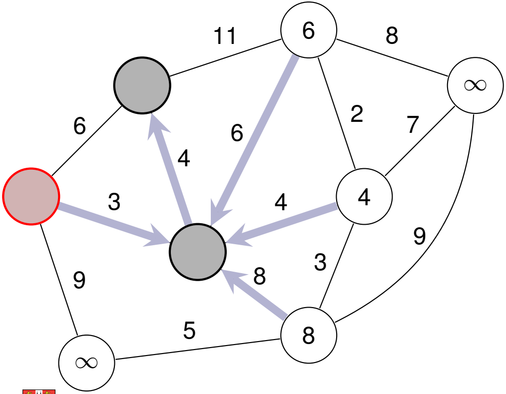

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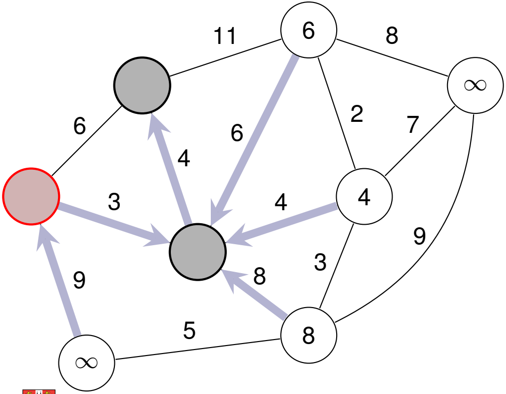

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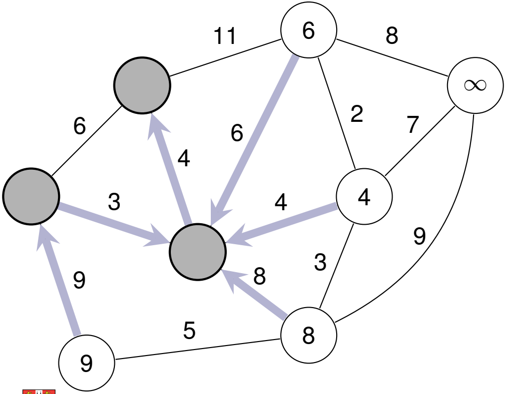

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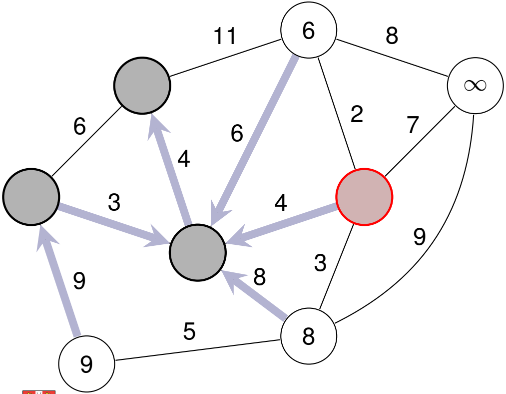

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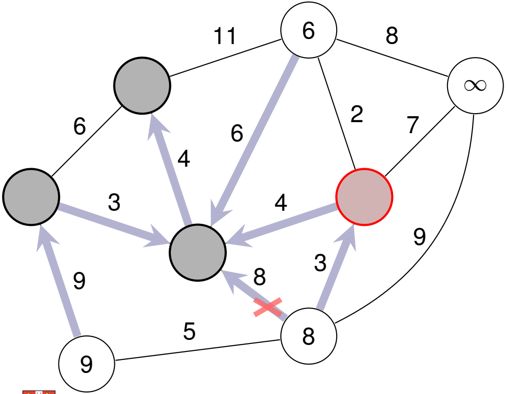

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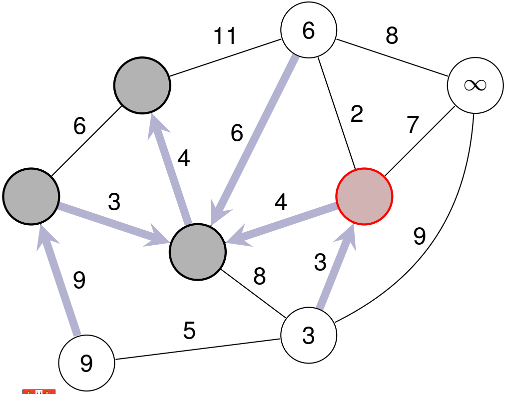

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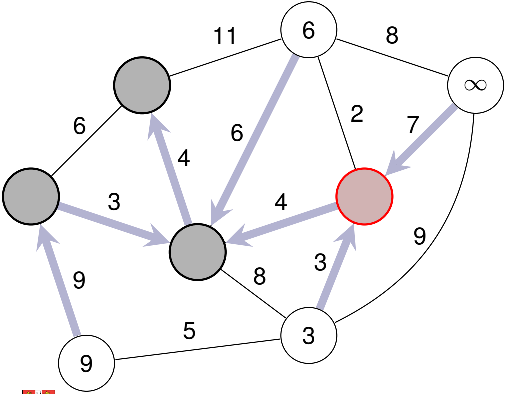

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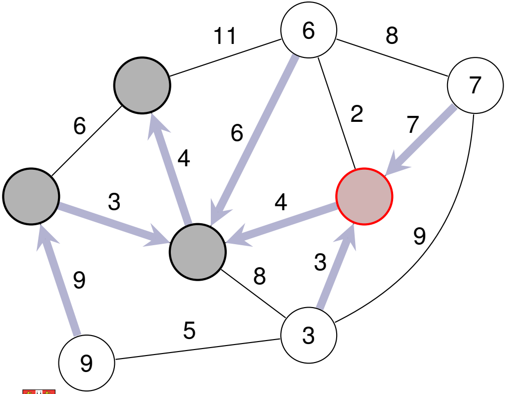

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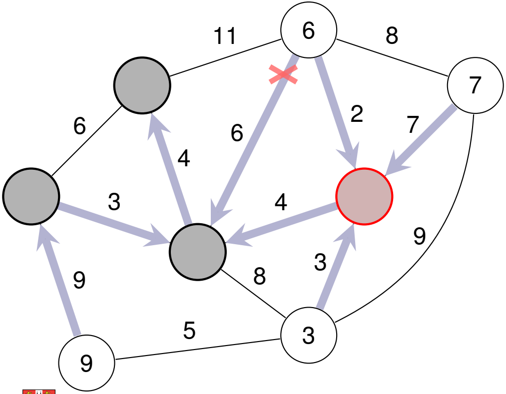

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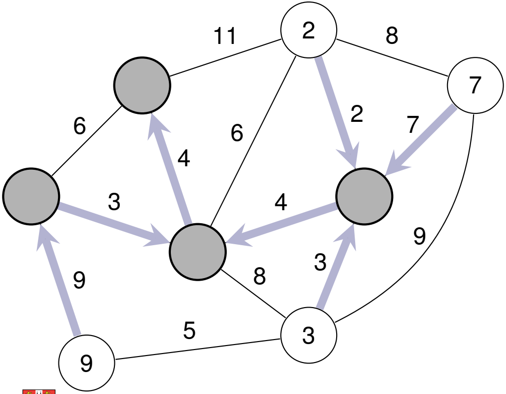

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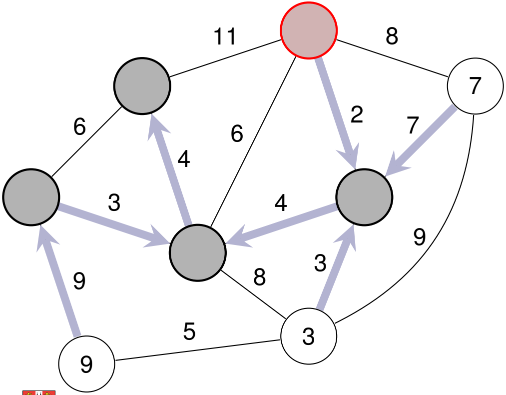

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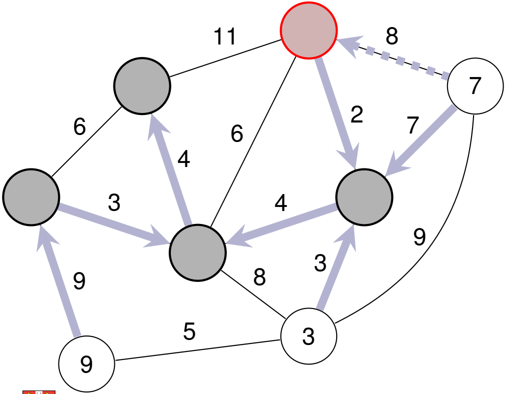

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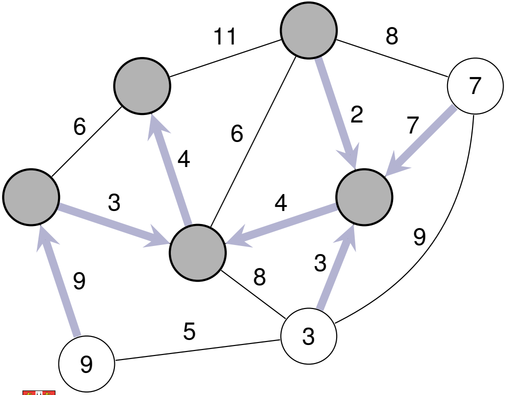

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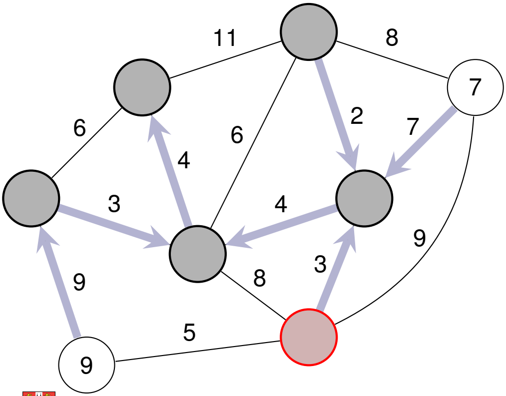

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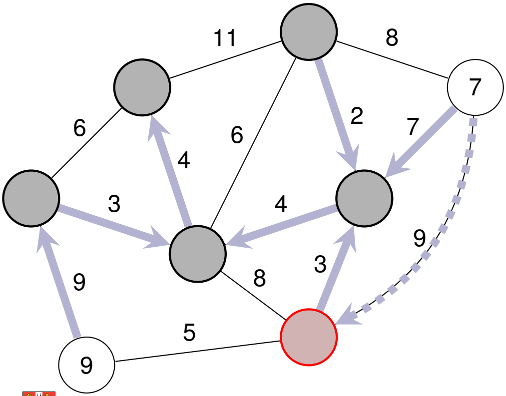

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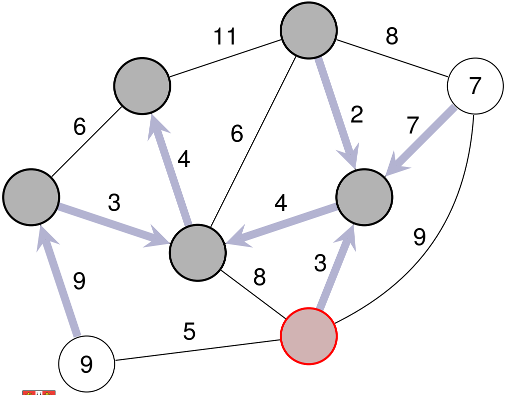

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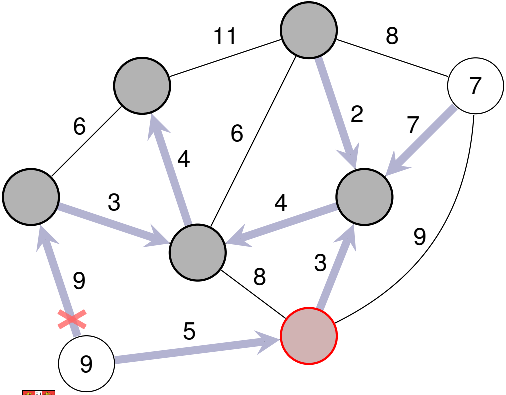

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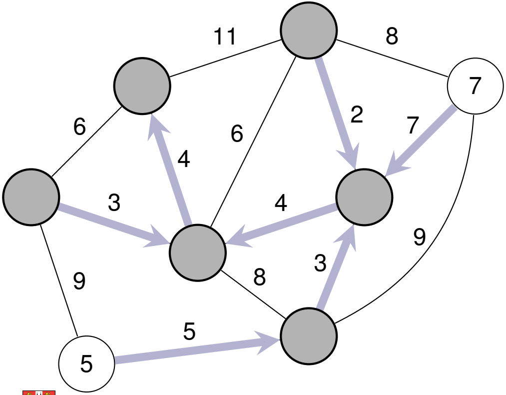

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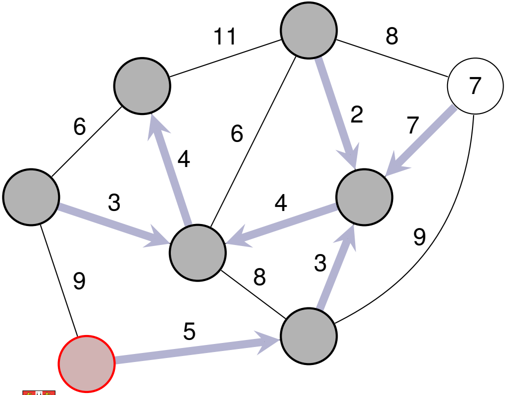

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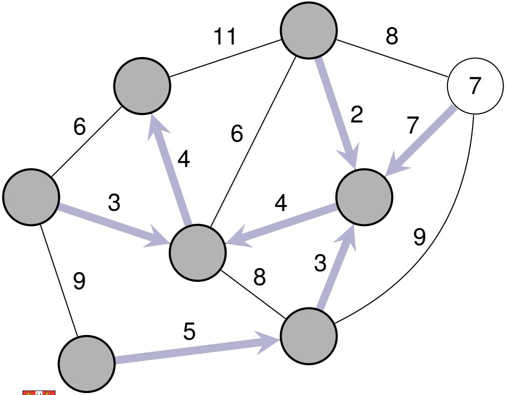

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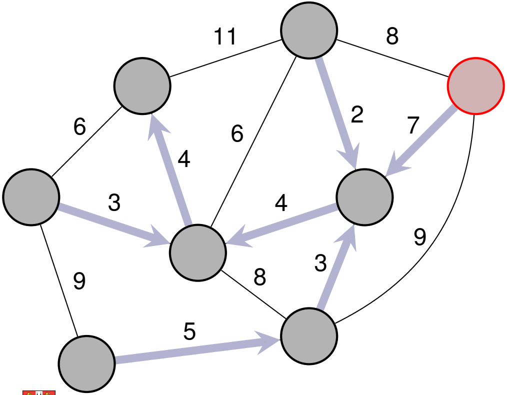

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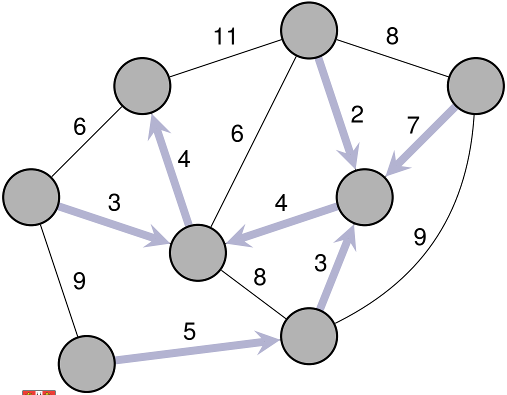

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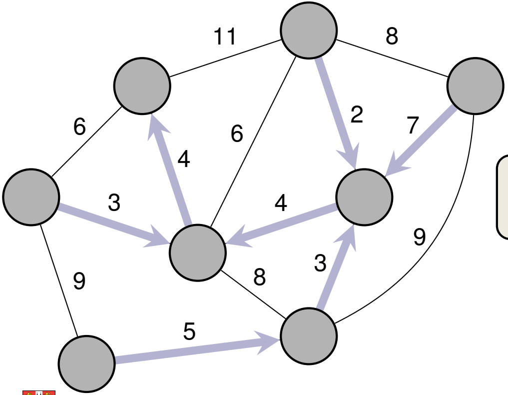


Final MST is given (implicitly) by the pointers!

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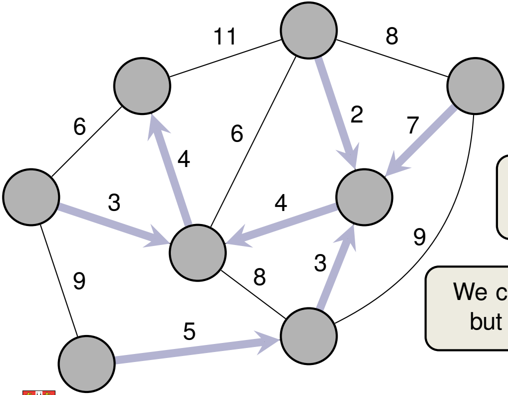
- Every vertex in $Q$ has key and pointer of least-weight edge to $V \setminus Q$
- At each step:
  1. extract vertex from $Q$ with smallest key $\Leftrightarrow$ safe edge of cut $(V \setminus Q, Q)$
  2. update keys and pointers of its neighbors in $Q$




Final MST is given (implicitly) by the pointers!

We computed same MST as Kruskal, but in a completely different order!

# Details of Prim's Algorithm

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- Add safe edge to the current MST as long as possible
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- Gradually transforms a forest into a MST by merging trees
- invokes disjoint set data structure
- Runtime $\mathcal{O}(V + E \log V)$

**Prim's Algorithm**

- Gradually extends a tree into a MST by adding incident edges
- invokes Fibonacci heaps (priority queue)
- Runtime $\mathcal{O}(V \log V + E)$

## Reverse-Delete Algorithm

Basic Idea

- Let $A$ be initially the set of all edges
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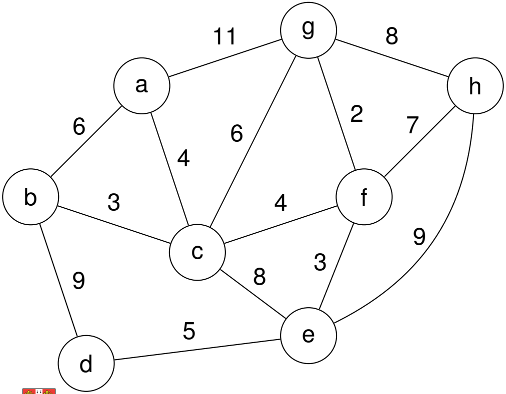

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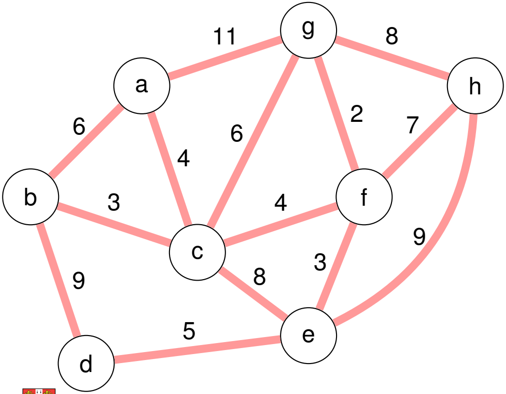

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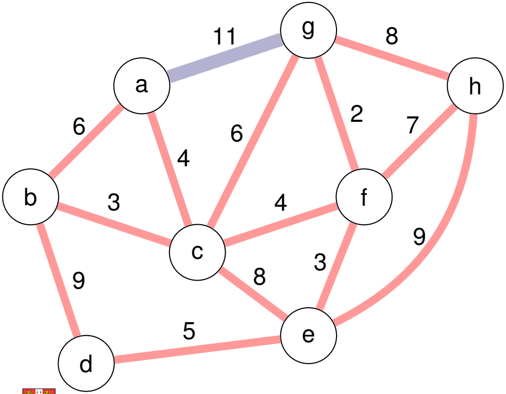

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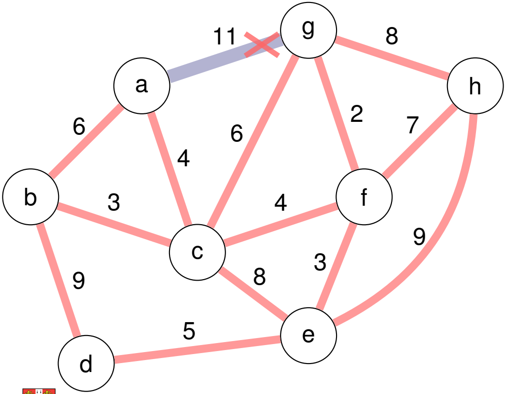

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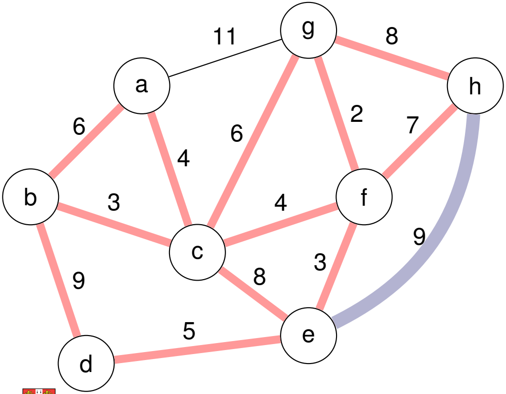

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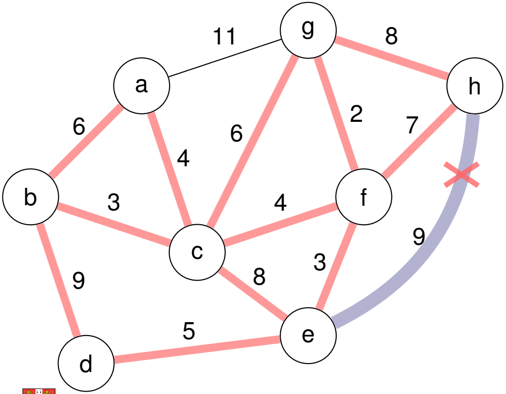

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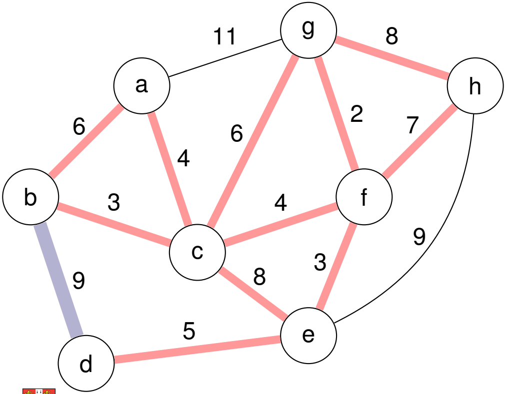

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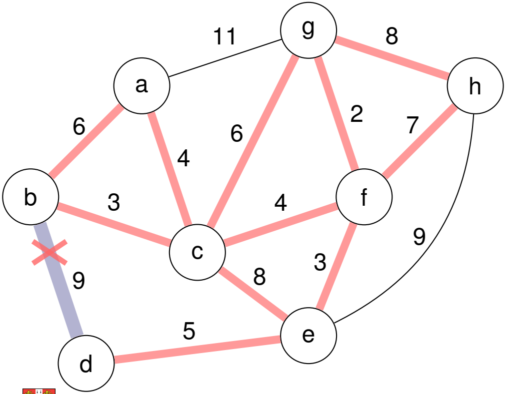

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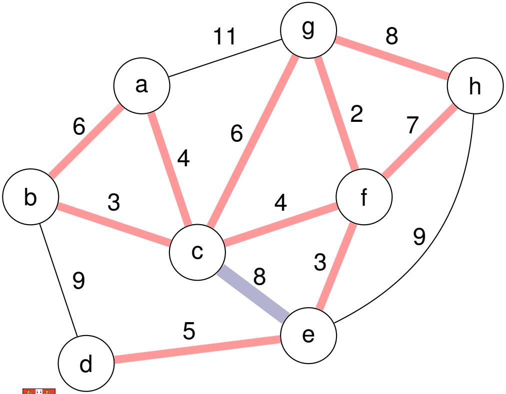

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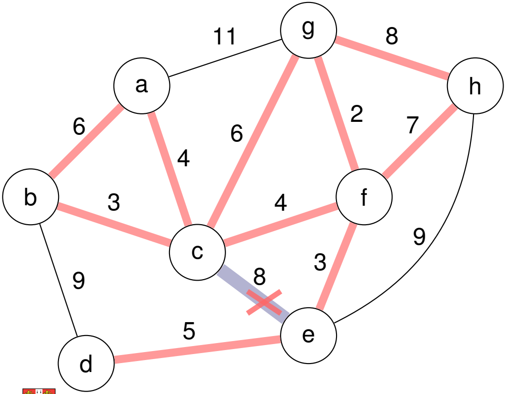

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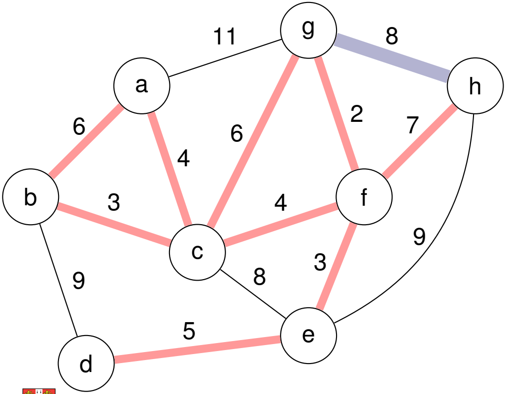

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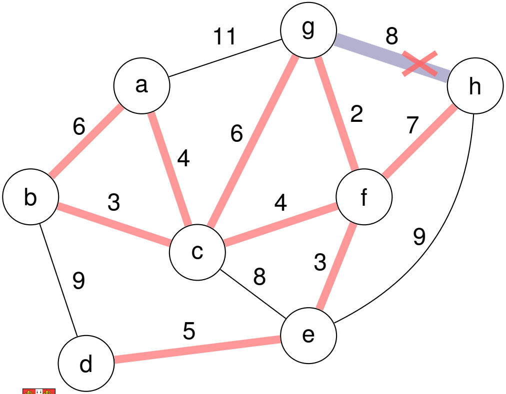

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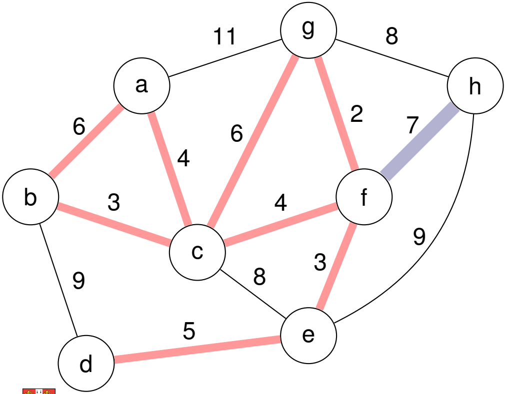

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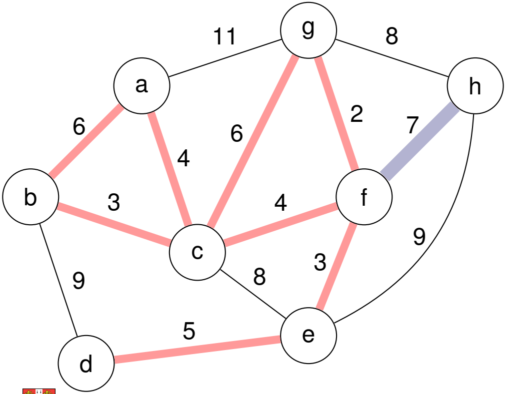

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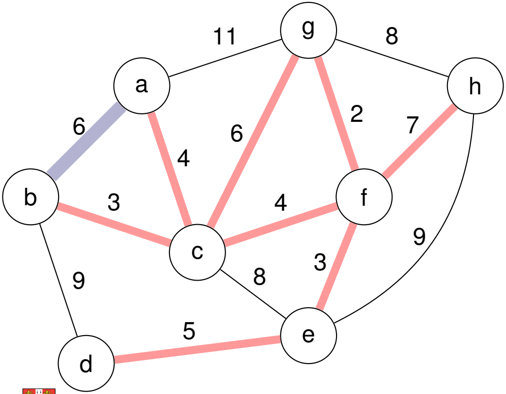

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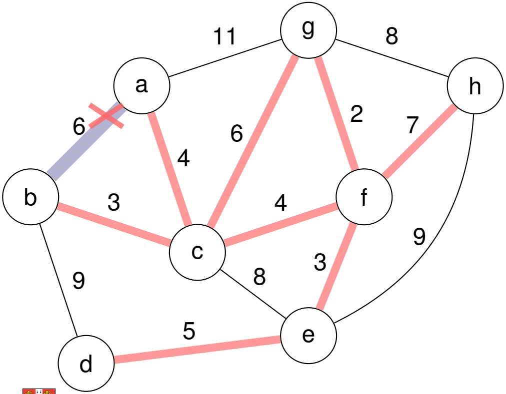

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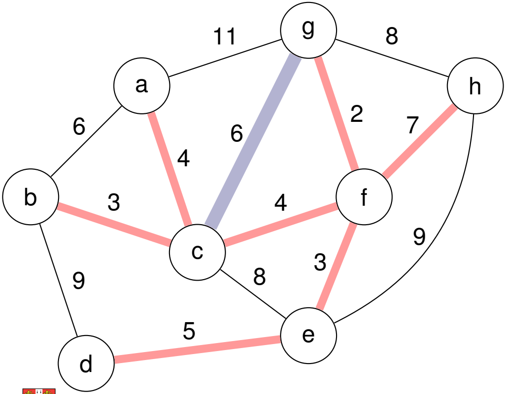

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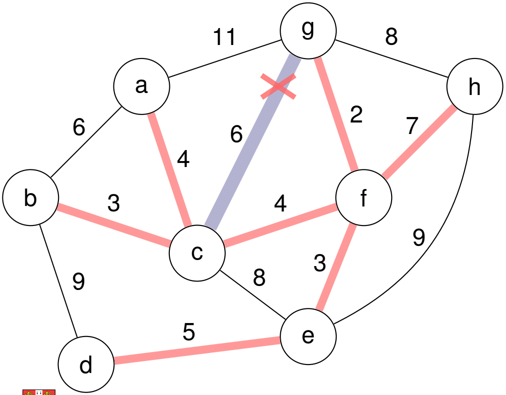

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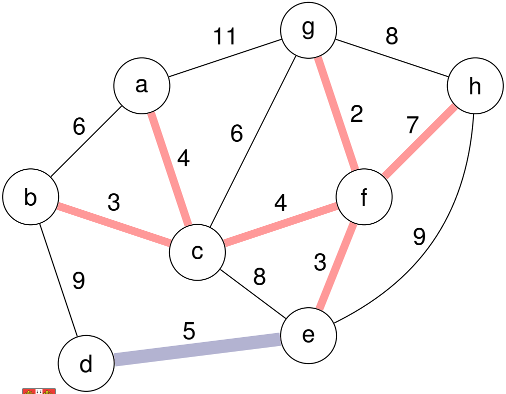

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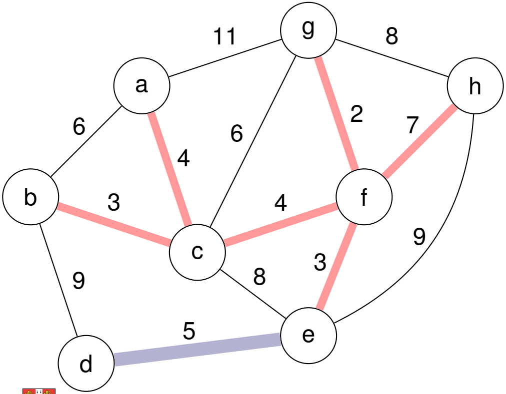

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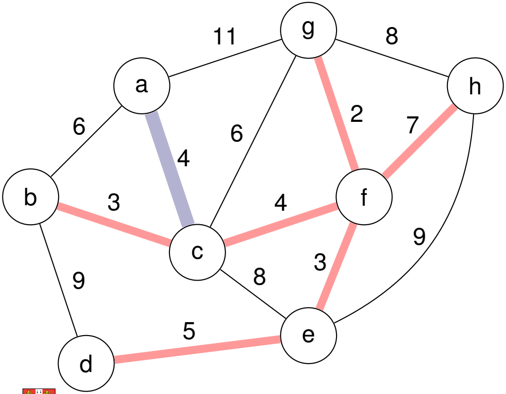

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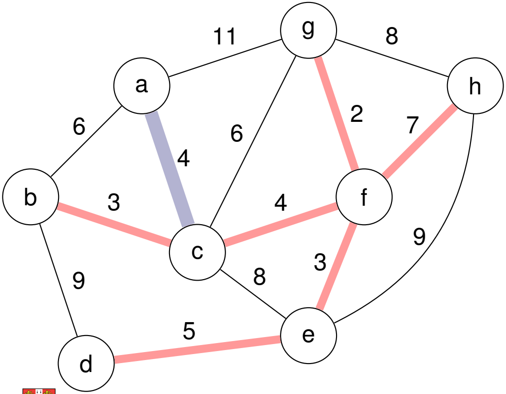

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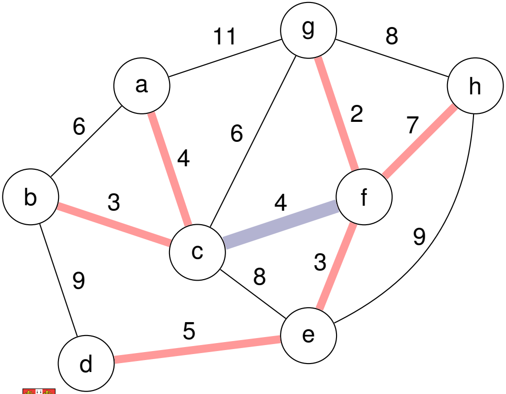

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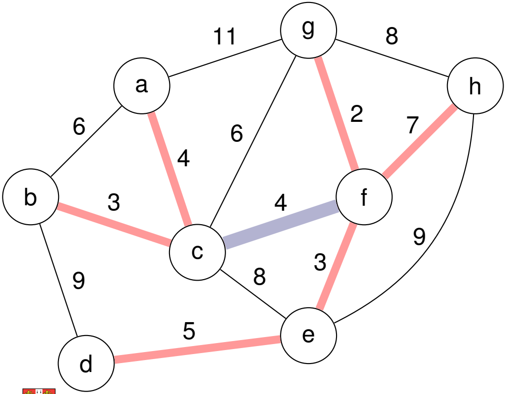

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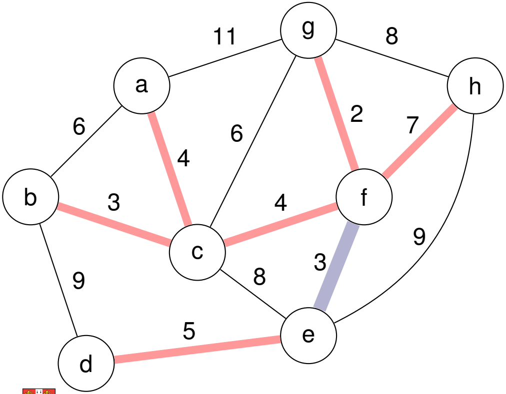

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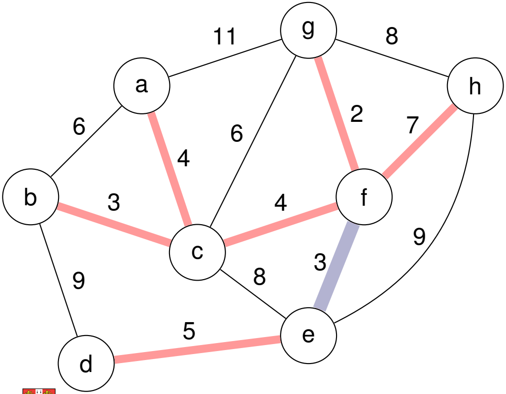

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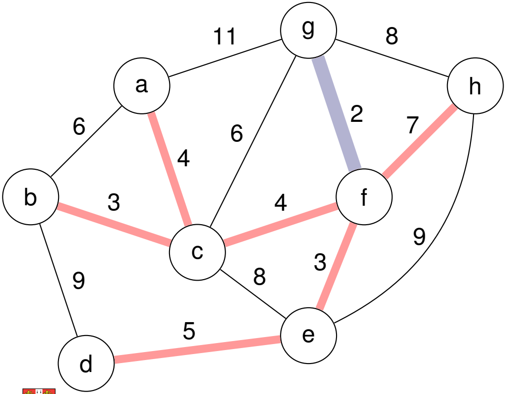

## Reverse-Delete Algorithm

Basic Idea

- Let $A$ be initially the set of all edges
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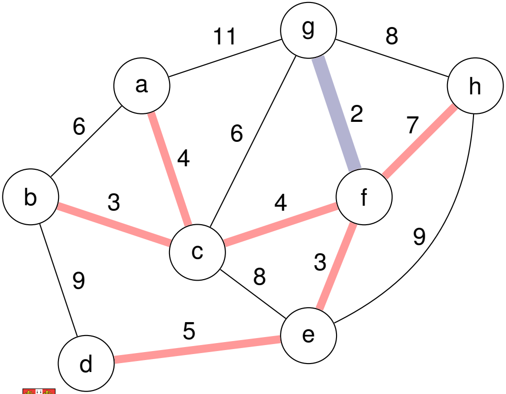

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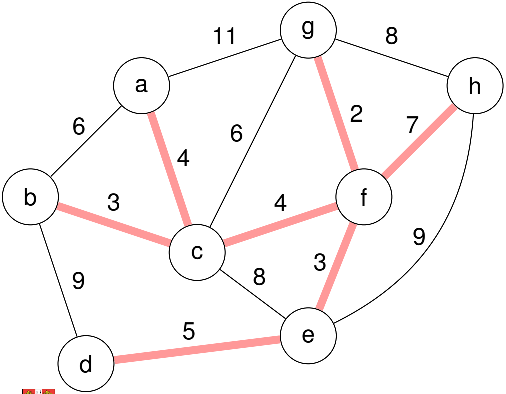

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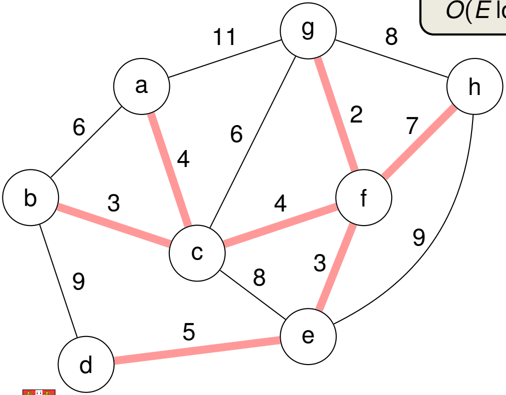
Can be implemented in time $O(E \log V (\log \log V)^3)$. [Thorup, 2000]



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Pettie, Ramachandran, JACM'2002

- deterministic MST algorithm with asymptotically optimal runtime
- however, the runtime itself is not known...