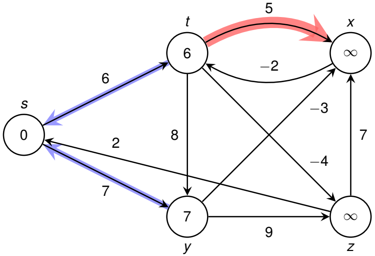


6.4: Single-Source Shortest Paths

Frank Stajano Thomas Sauerwald

Lent 2015

UNIVERSITY OF CAMBRIDGE

Outline

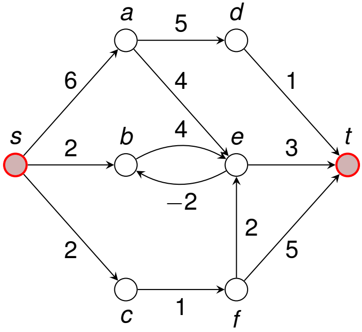
Introduction

Bellman-Ford Algorithm

Shortest Path Problem

Shortest Path Problem

- Given: directed graph $G = (V, E)$ with edge weights, pair of vertices $s, t \in V$



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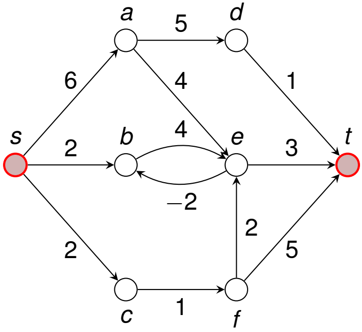

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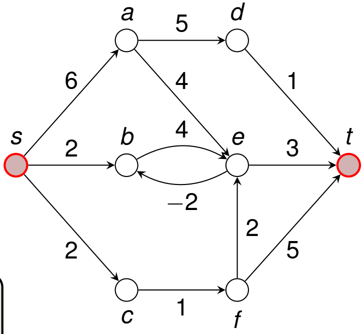

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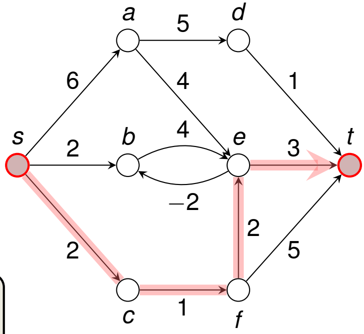

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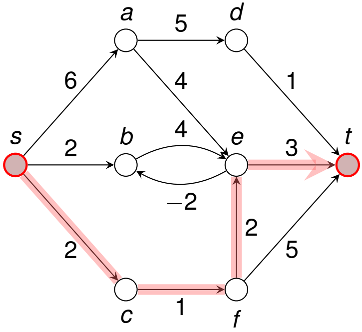

How to cope with an **unweighted** graph G?

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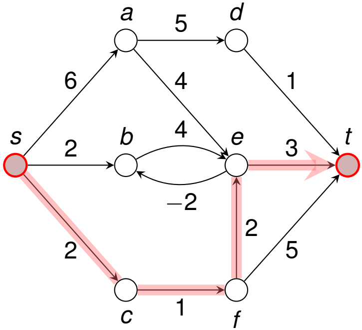

How to cope with an **unweighted** graph G?

Two possible answers are:

1. Run BFS (computes shortest paths in unweighted graphs)
2. Add a weight of 1 to all edges

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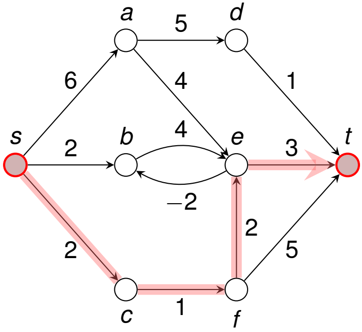


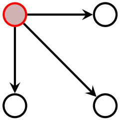
Applications

- Car Navigation, Traffic Planning, Internet Routing, Arbitrage in Concurrency Exchange, . . .

Variants of Shortest Path Problems

Single-source shortest-paths problem (SSSP)

- Bellman-Ford Algorithm
- Dijsktra Algorithm



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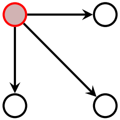
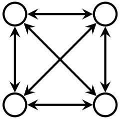
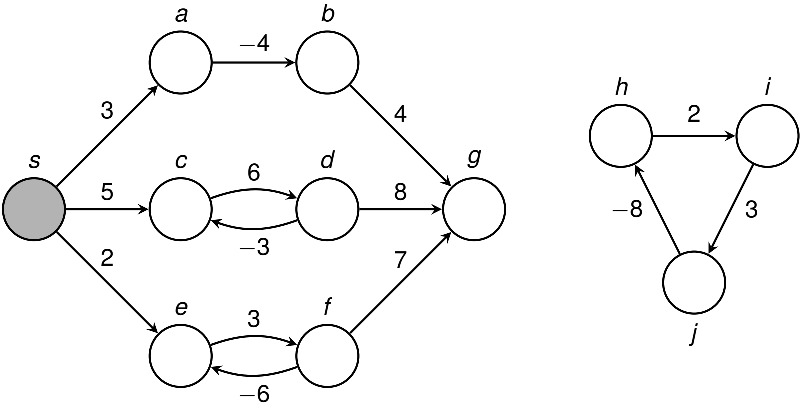


All-pairs shortest-paths problem (APSP)

- Shortest Paths via Matrix Multiplication
- Johnson’s Algorithm



Distances and Negative-Weight Cycles (Figure 24.1)



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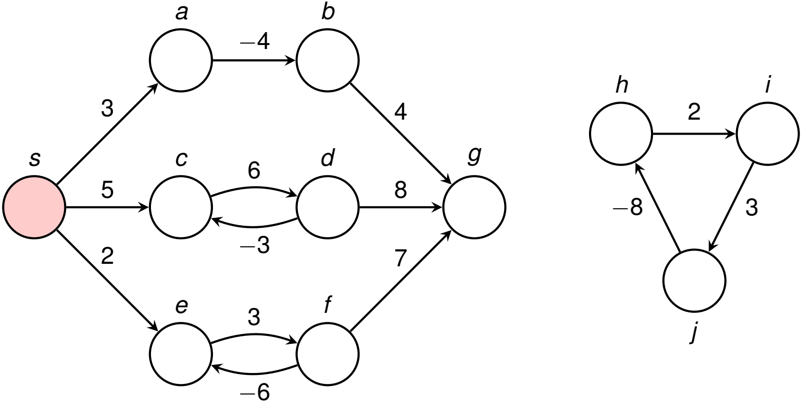

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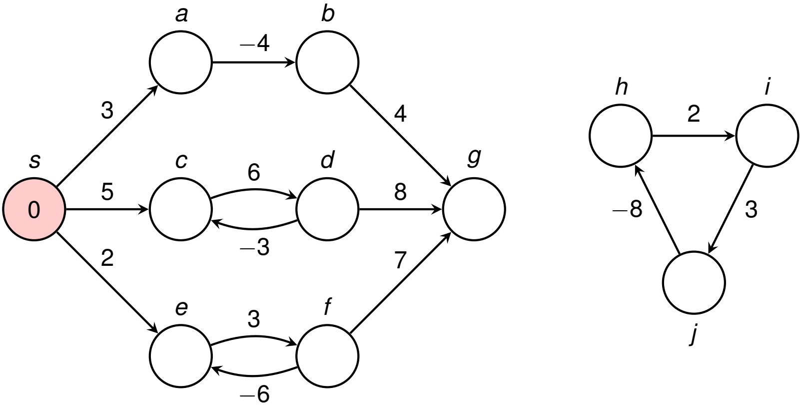

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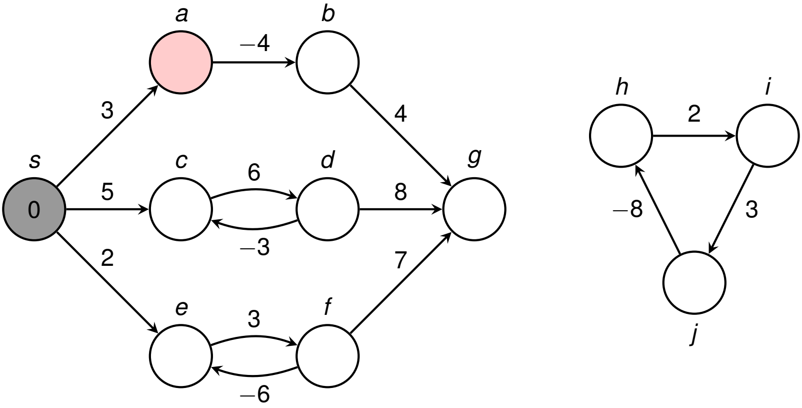

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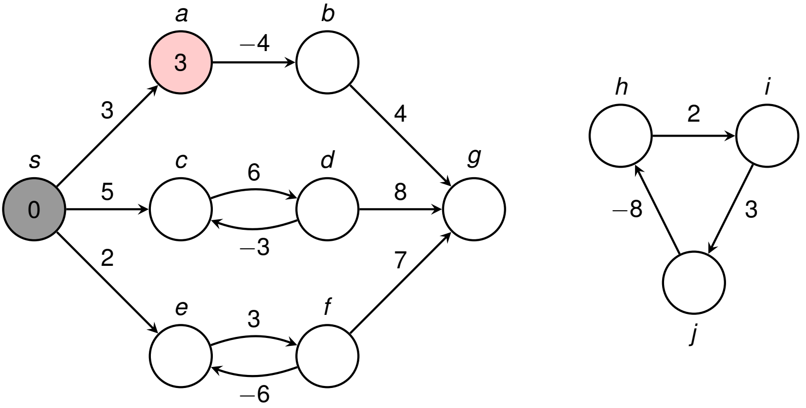

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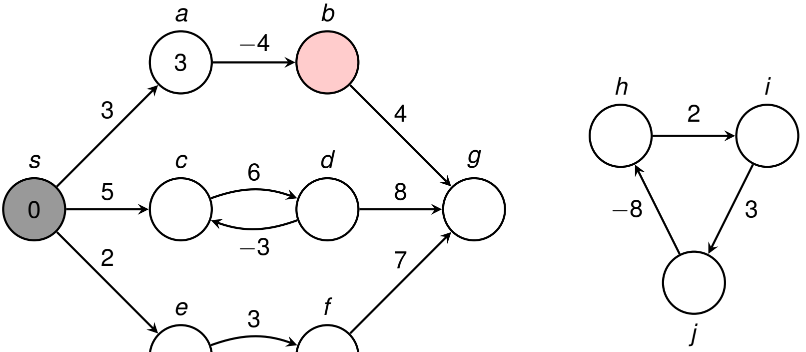

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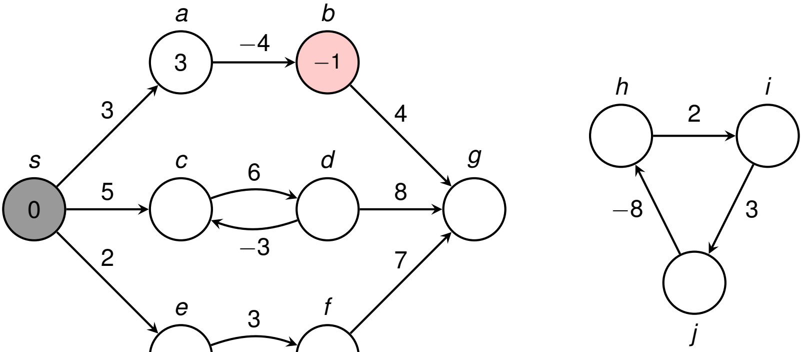

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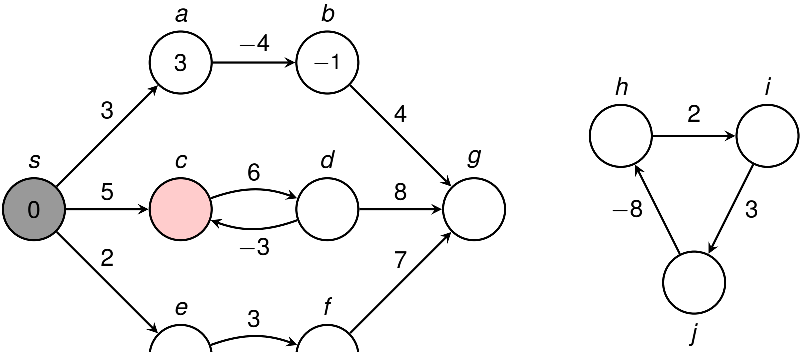

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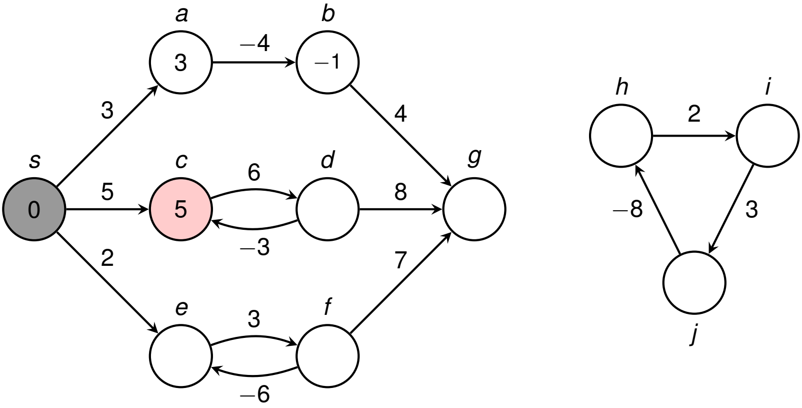

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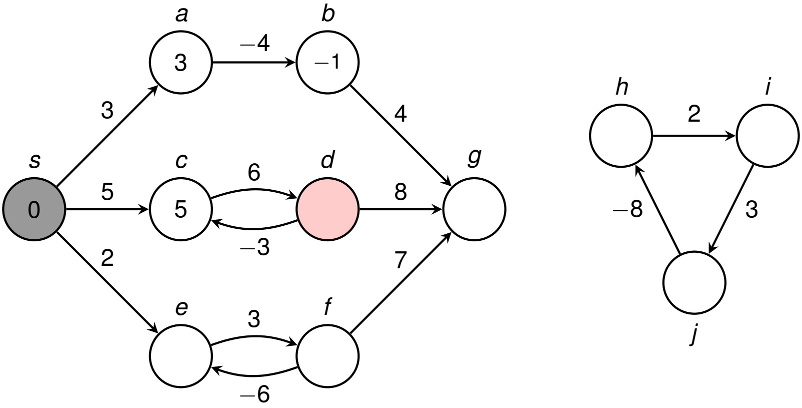

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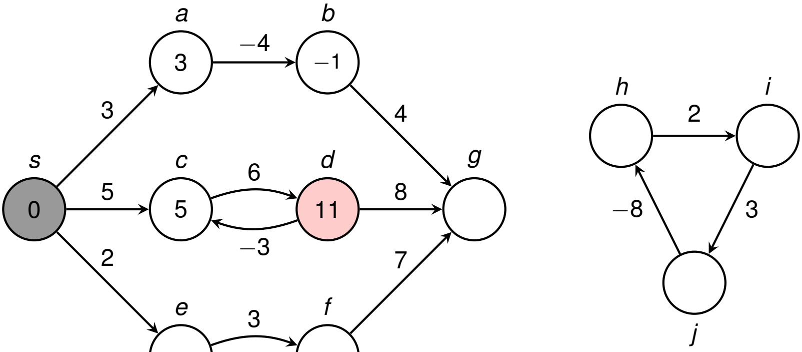

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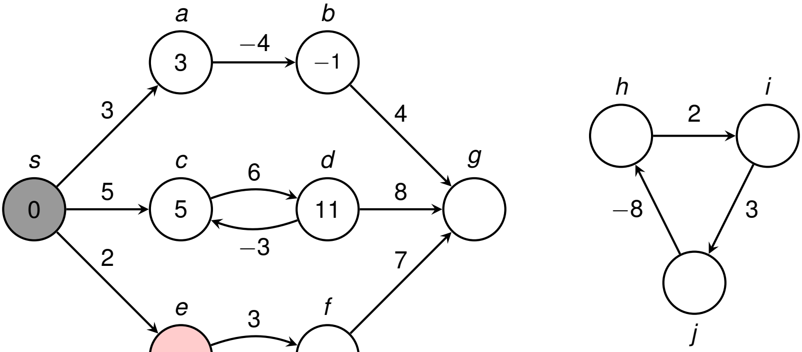

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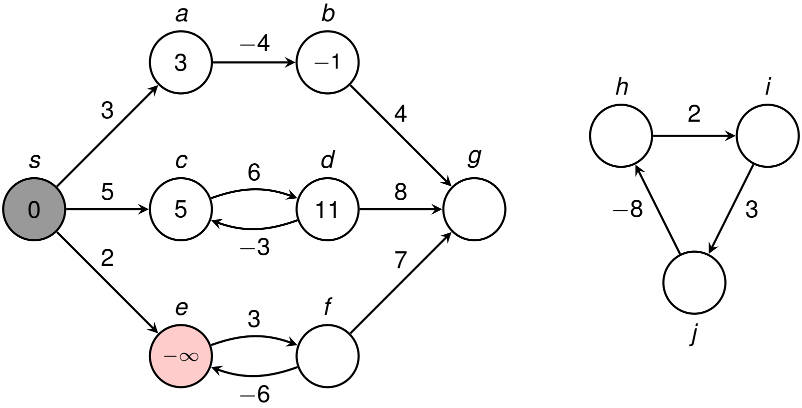

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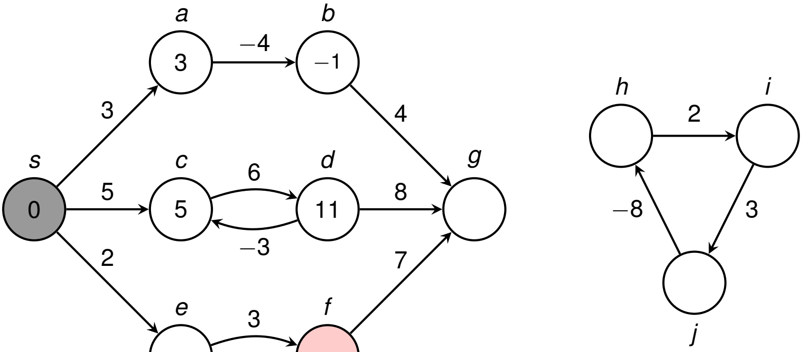

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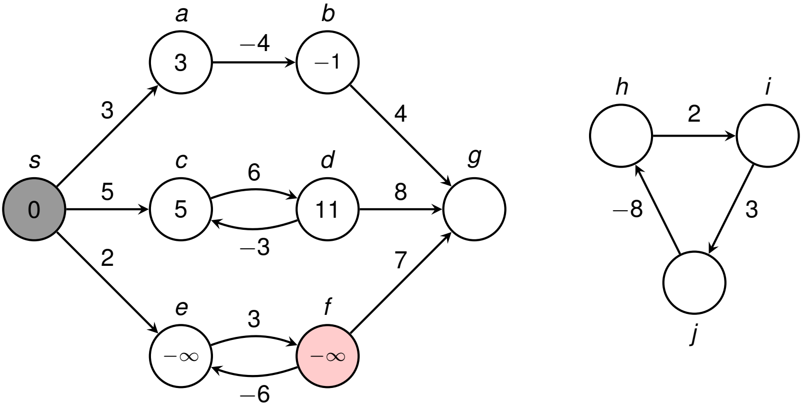

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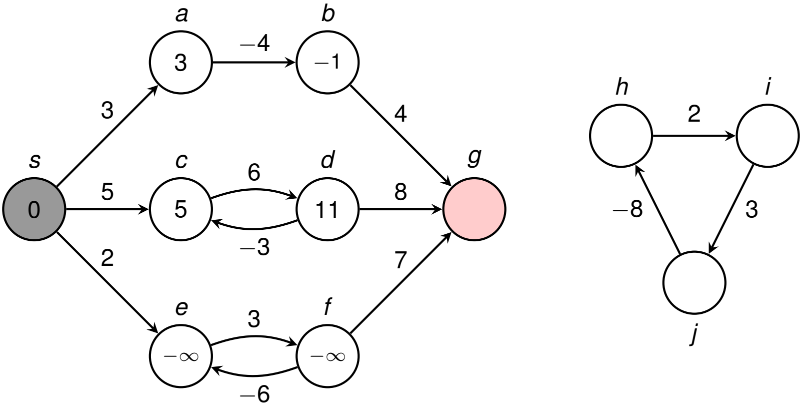

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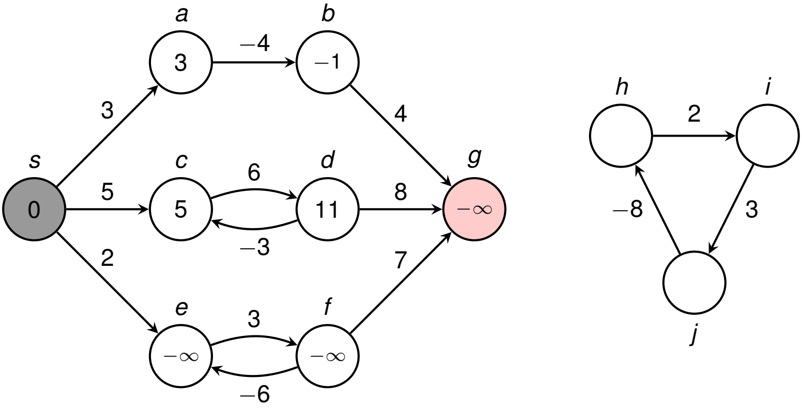

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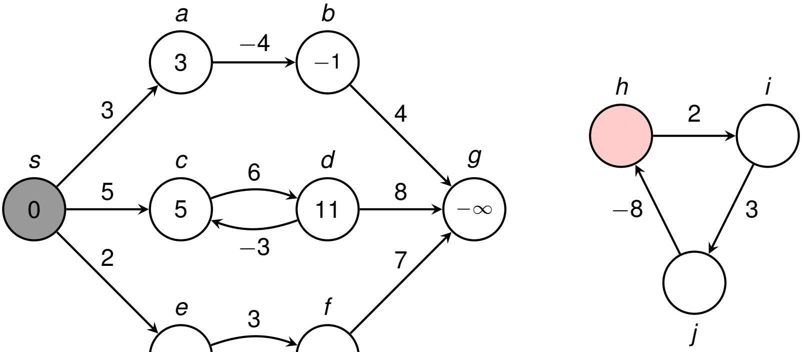

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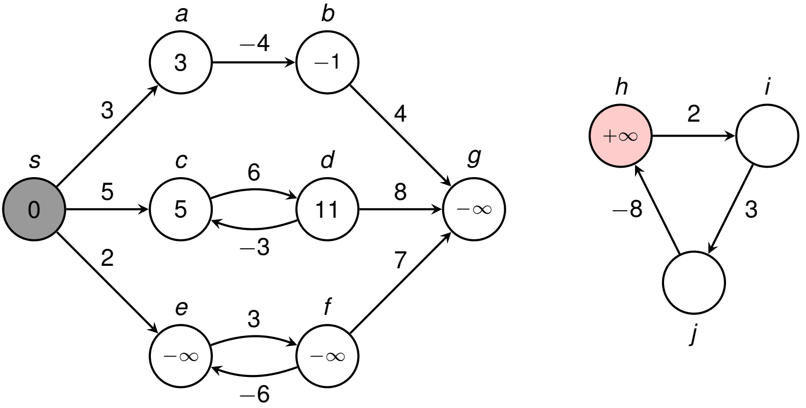

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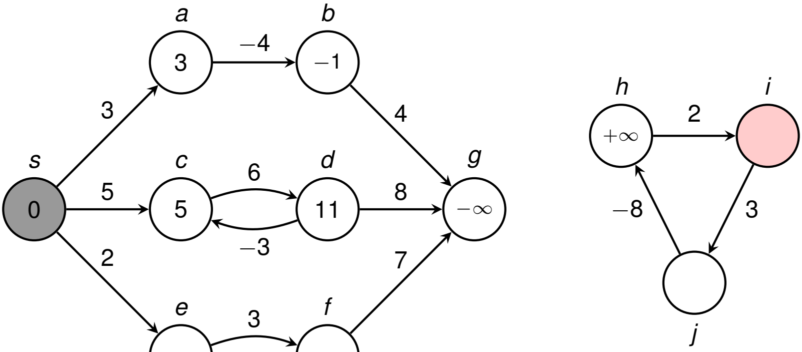

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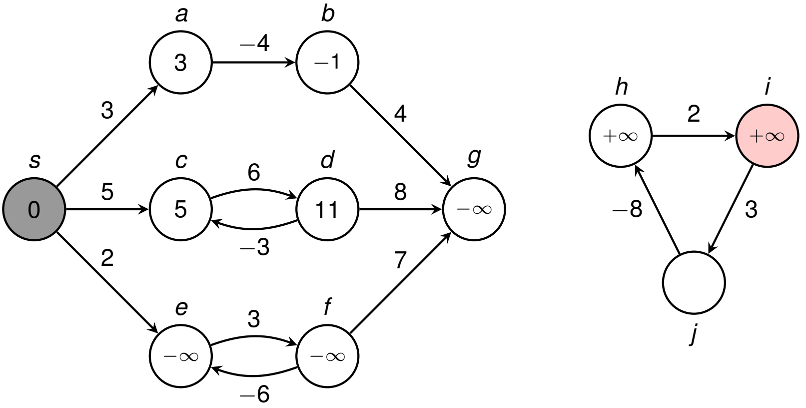

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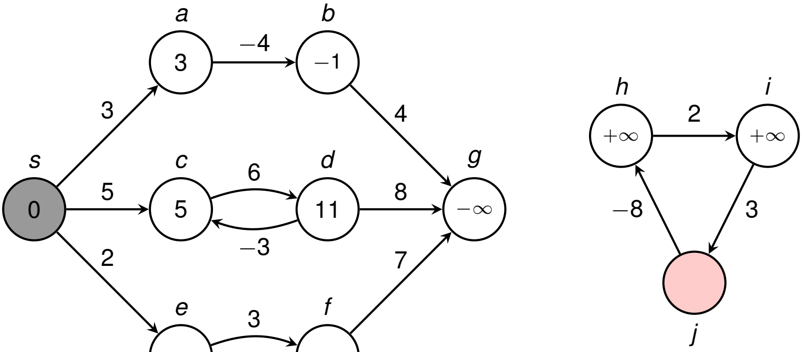

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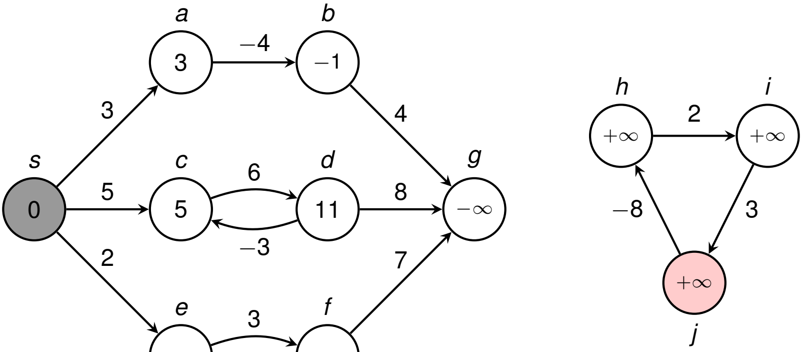

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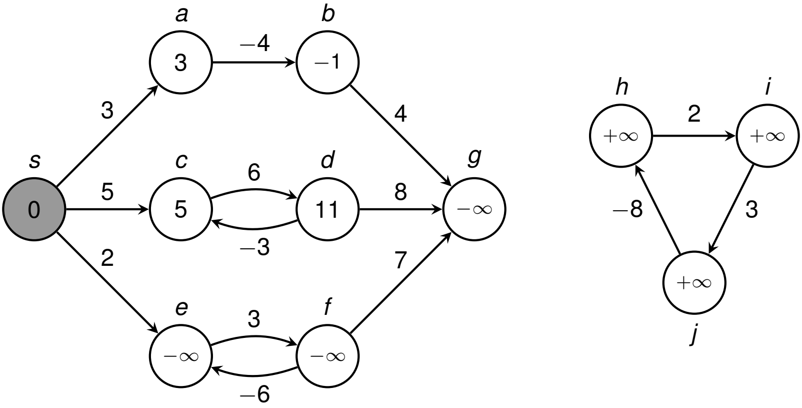

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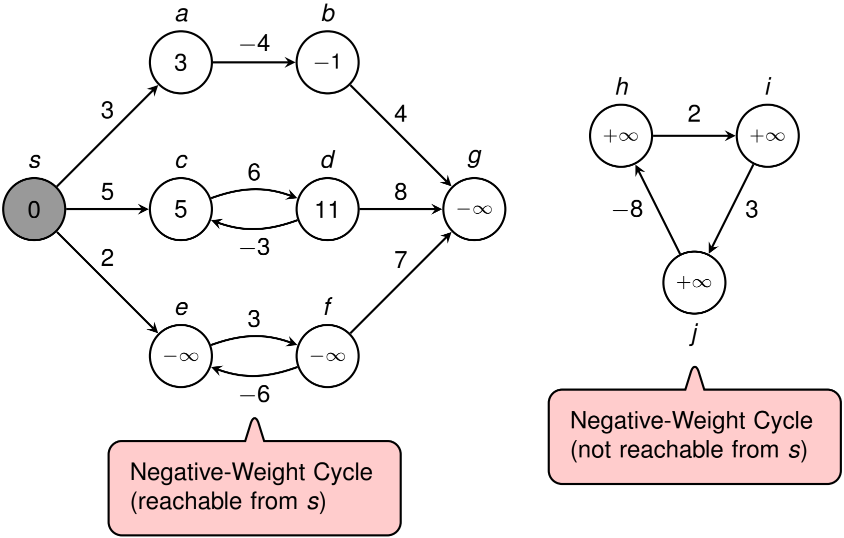

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Fix the source vertex $s \in V$

- $v.\delta$ is the length of the shortest path (distance) from s to v
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Given estimates $u.d$ and $v.d$, can we find a better path from v using the edge (u, v)?

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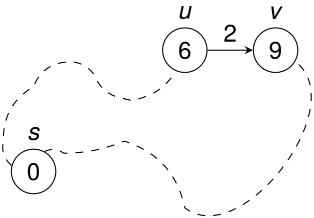

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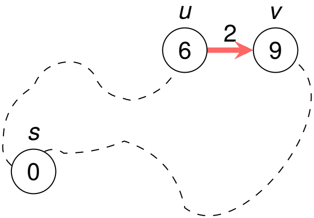

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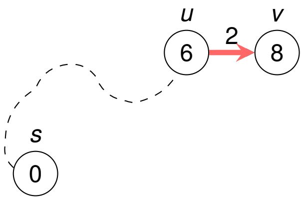

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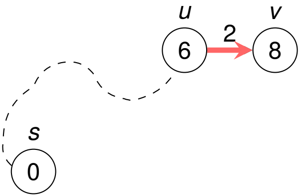


After relaxing (u, v), regardless of whether we found a shortcut: $v.d \leq u.d + w(u, v)$

Properties of Shortest Paths and Relaxations

Toolkit

Triangle inequality (Lemma 24.10)

- For any edge $(u, v) \in E$, we have $v.\delta \leq u.\delta + w(u, v)$

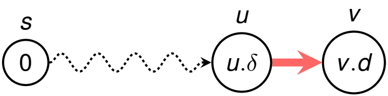
Upper-bound Property (Lemma 24.11)

- We always have $v.d \geq v.\delta$ for all $v \in V$, and once $v.d$ achieves the value $v.\delta$, it never changes.

Convergence Property (Lemma 24.14)

- If $s \rightsquigarrow u \rightarrow v$ is a shortest path from s to v, and if $u.d = u.\delta$ prior to relaxing edge (u, v), then $v.d = v.\delta$ at all times afterward.

$v.d$



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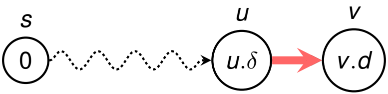

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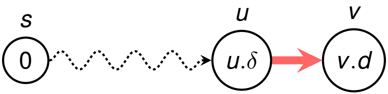

$$\begin{aligned} v.d &\leq u.d + w(u, v) \\ &= u.\delta + w(u, v) \end{aligned}$$

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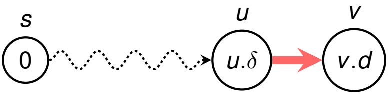

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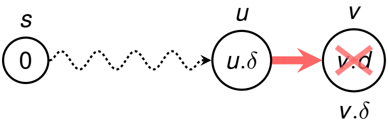

$$
\begin{aligned}
v.d &\leq u.d + w(u, v) \\
&= u.\delta + w(u, v) \\
&= v.\delta
\end{aligned}
$$

Since $v.d \geq v.\delta$, we have $v.d = v.\delta$. $\square$

Path-Relaxation Property

Path-Relaxation Property (Lemma 24.15)

If $p = (v_0, v_1, \ldots, v_k)$ is a shortest path from $s = v_0$ to v_k, and we relax the edges of p in the order $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = v_k.\delta$ (regardless of the order of other relaxation steps).

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Proof:

- By induction on i, $0 \leq i \leq k$:
  After the ith edge of p is relaxed, we have $v_i.d = v_i.\delta$.

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  After the ith edge of p is relaxed, we have $v_i.d = v_i.\delta$.
- For $i = 0$, by the initialization $s.d = s.\delta = 0$.
  Upper-bound Property $\Rightarrow$ the value of $s.d$ never changes after that.

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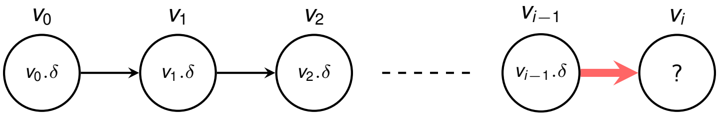

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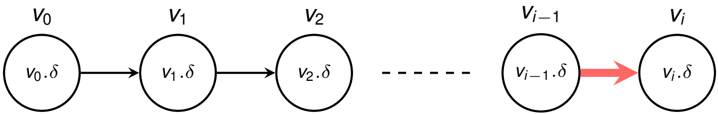

Path-Relaxation Property

"Propagation": By relaxing proper edges, set of vertices with $v.\delta = v.d$ gets larger

Path-Relaxation Property (Lemma 24.15)

If $p = (v_0, v_1, \ldots, v_k)$ is a shortest path from $s = v_0$ to v_k, and we relax the edges of p in the order $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = v_k.\delta$ (regardless of the order of other relaxation steps).

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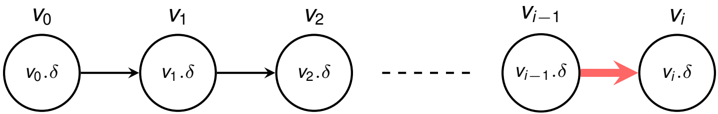

The Bellman-Ford Algorithm

```
BELLMAN-FORD(G,w,s)
assert(s in G.vertices())
for v in G.vertices()
   v.predecessor = None
   v.d = Infinity
s.d = 0

repeat |V|-1 times
   for e in G.edges()
   Relax edge e=(u,v):  Check if u.d + w(u,v) < v.d
   if e.start.d + e.weight.d < e.end.d:
       e.end.d = e.start.d + e.weight
       e.end.predecessor = e.start

for e in G.edges()
    if e.start.d + e.weight.d < e.end.d:
       return FALSE
return TRUE
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Time Complexity

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10:       e.end.d = e.start.d + e.weight
11:       e.end.predecessor = e.start
12:
13: for e in G.edges()
14:    if e.start.d + e.weight.d < e.end.d:
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16: return TRUE
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Time Complexity

- A single call of line 9-11 costs $\mathcal{O}(1)$

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```

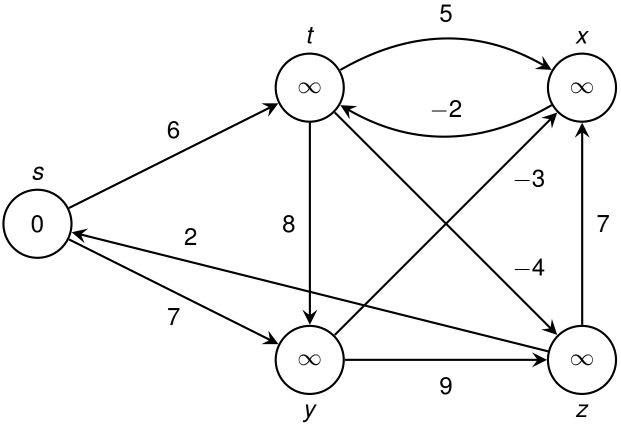
Time Complexity

- A single call of line 9-11 costs $\mathcal{O}(1)$
- In each pass every edge is relaxed $\Rightarrow \mathcal{O}(E)$ time per pass
- Overall $(V-1)+1 = V$ passes $\Rightarrow \mathcal{O}(V \cdot E)$ time

Complete Run of Bellman-Ford (Figure 24.4)

Pass: 1

Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



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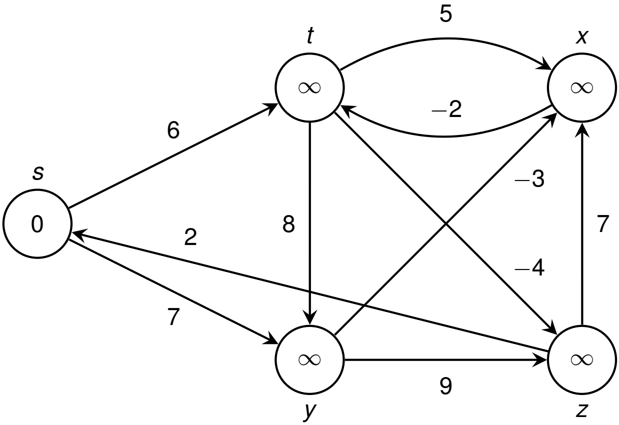

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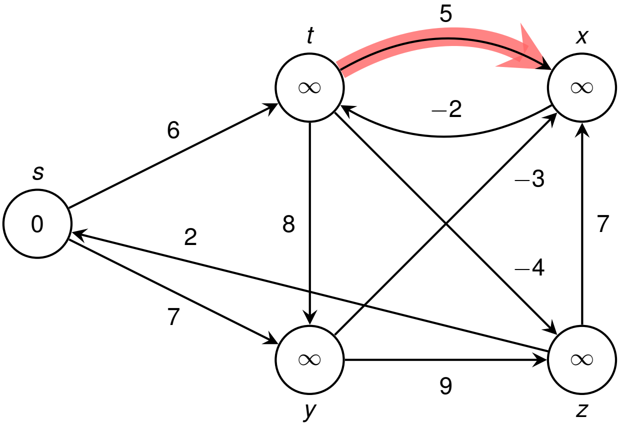

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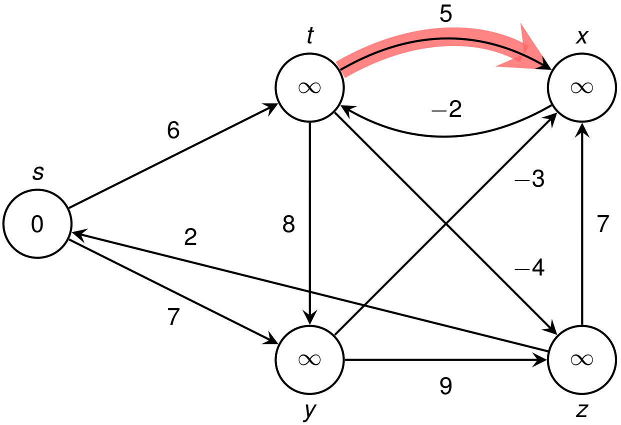

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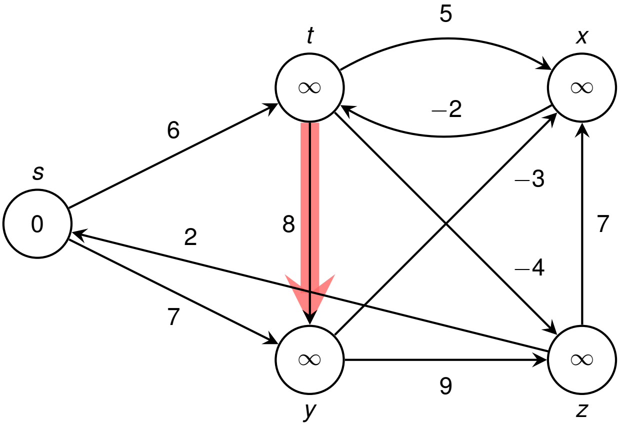

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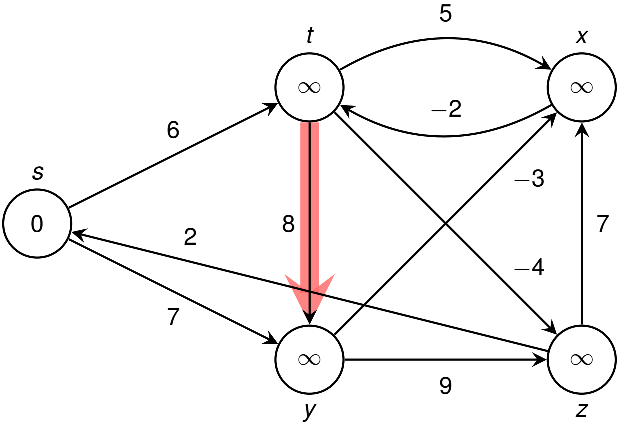

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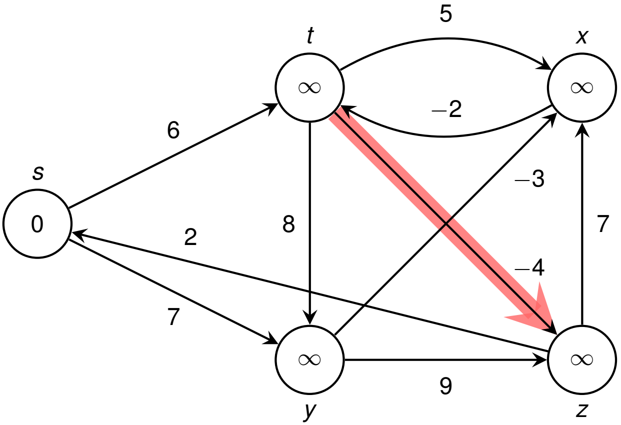

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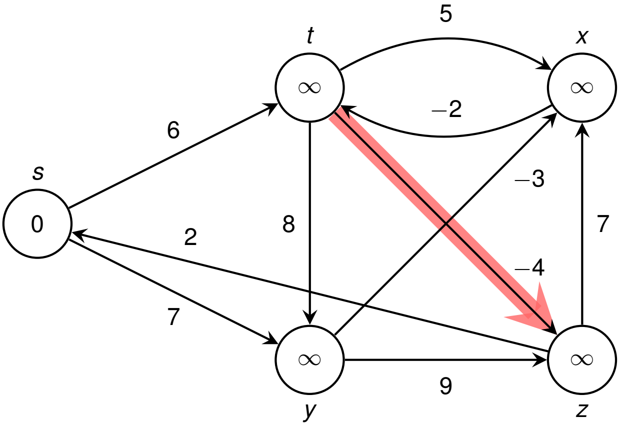

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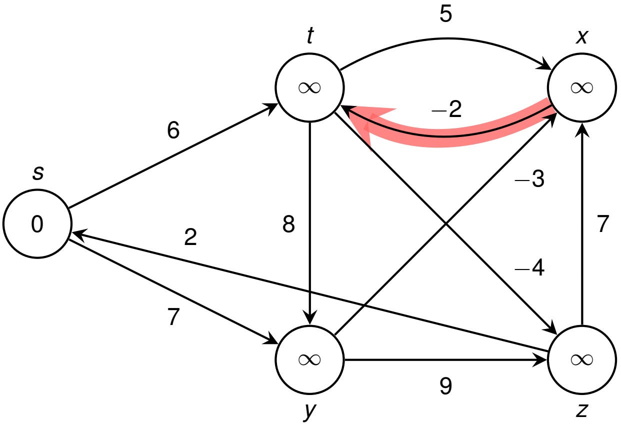

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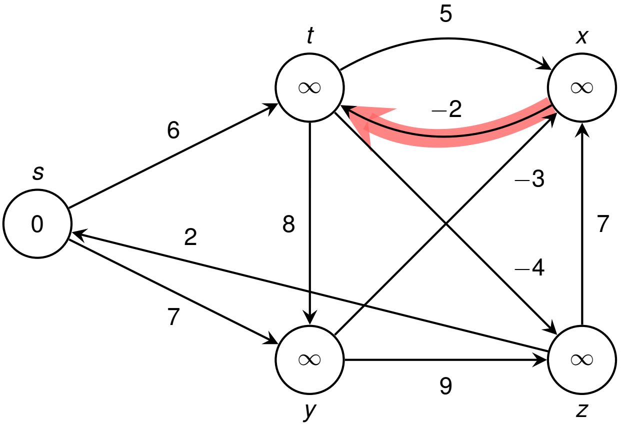

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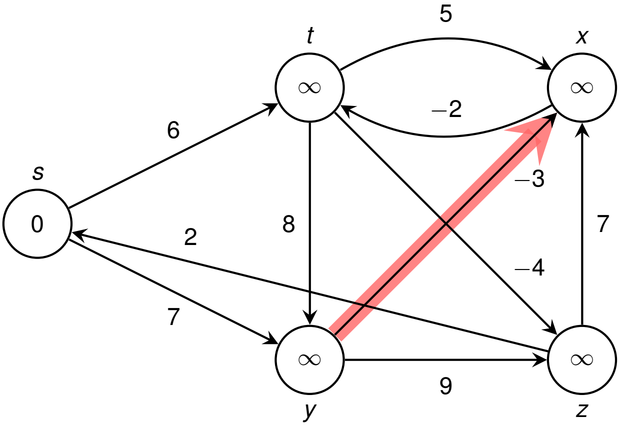

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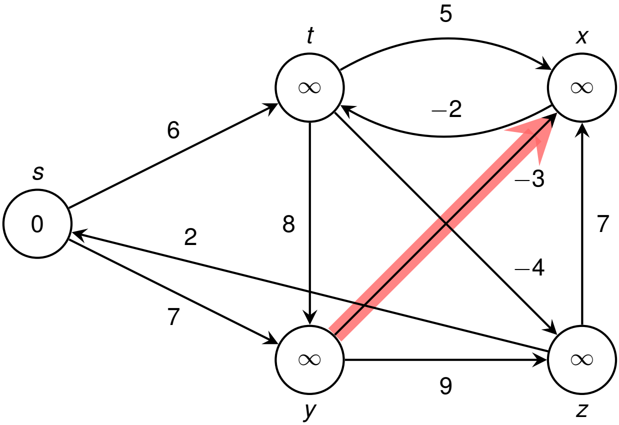

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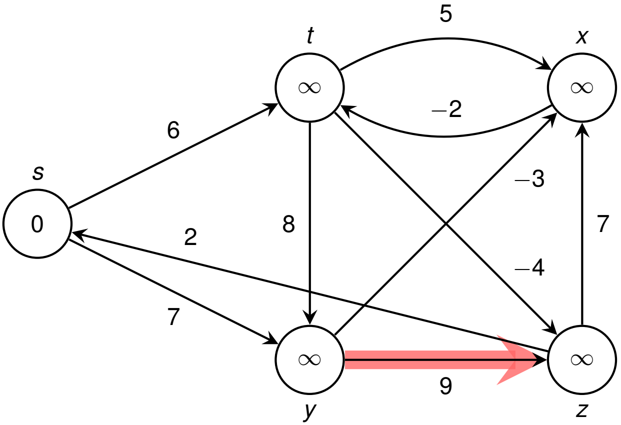

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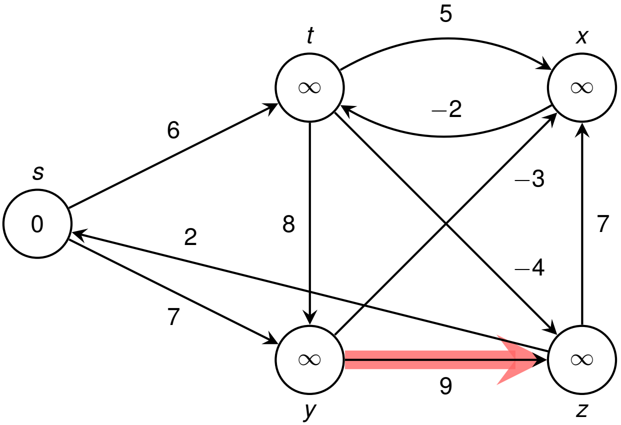

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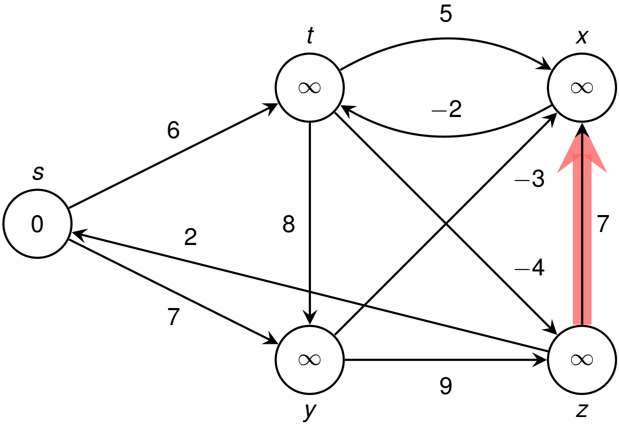

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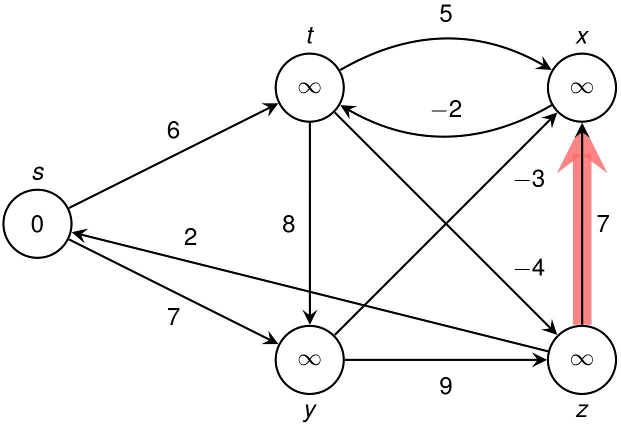

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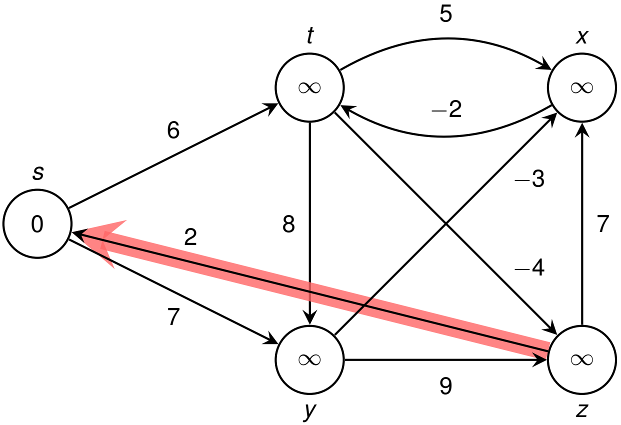

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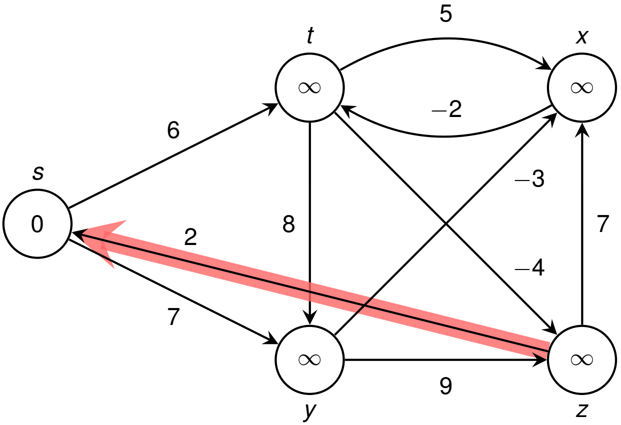

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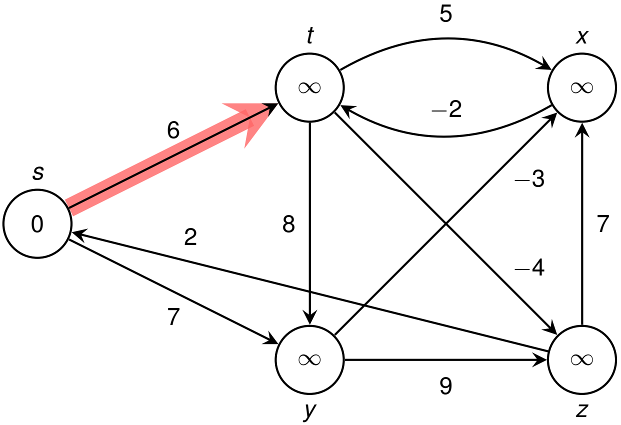

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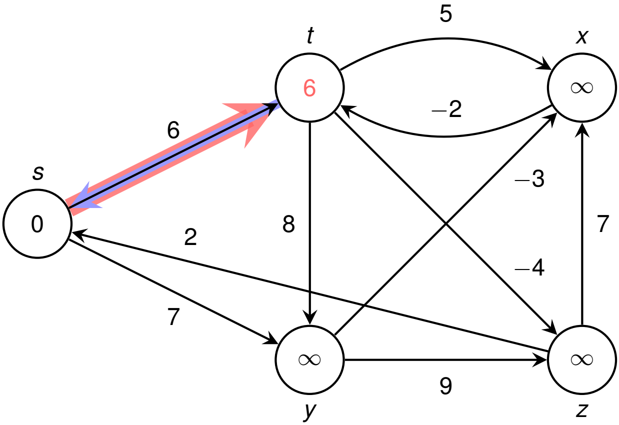

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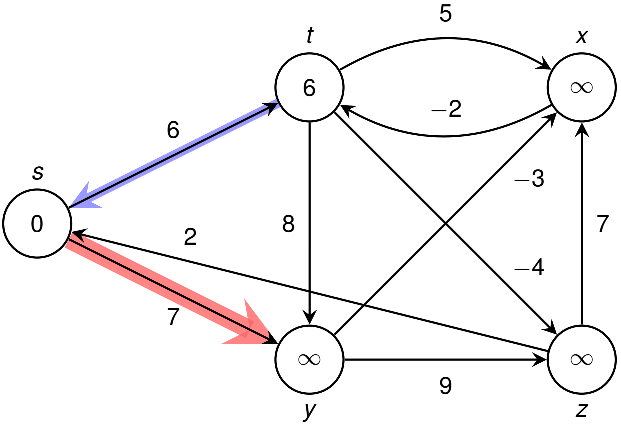

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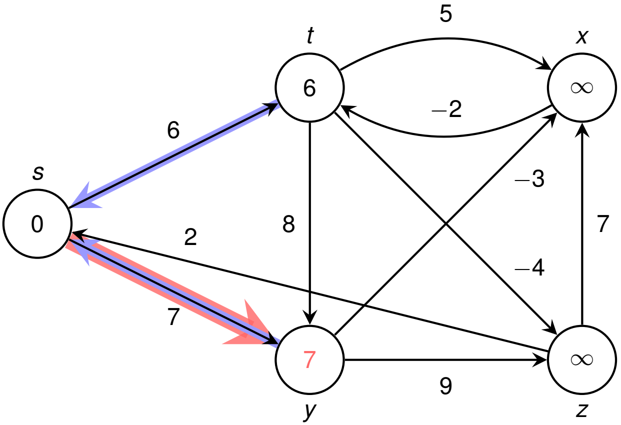
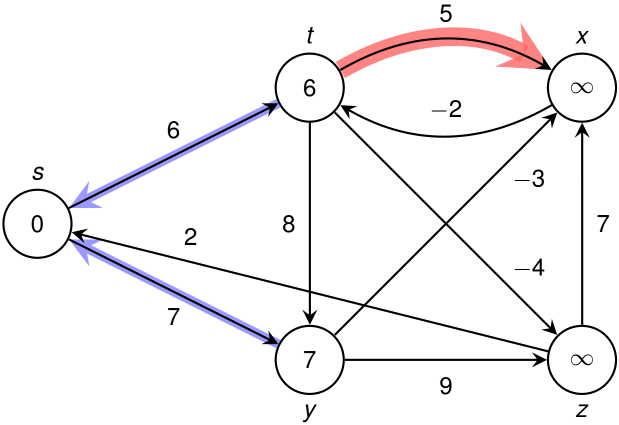


Complete Run of Bellman-Ford (Figure 24.4)

Pass: 2

Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



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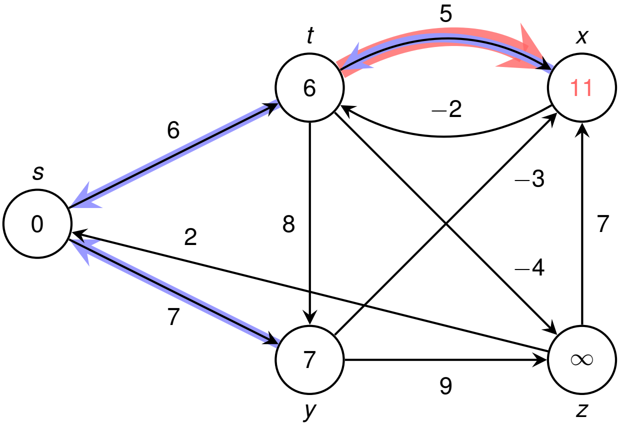

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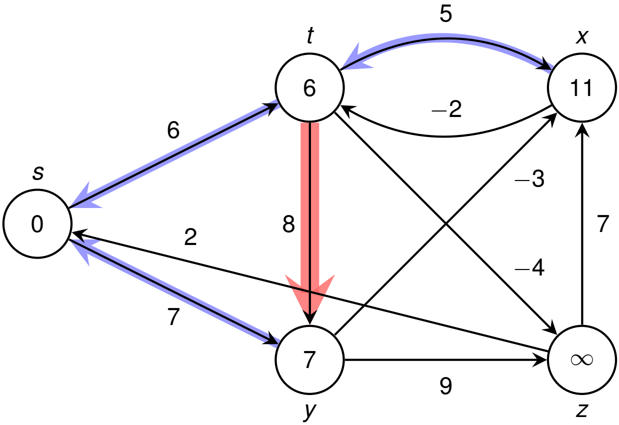

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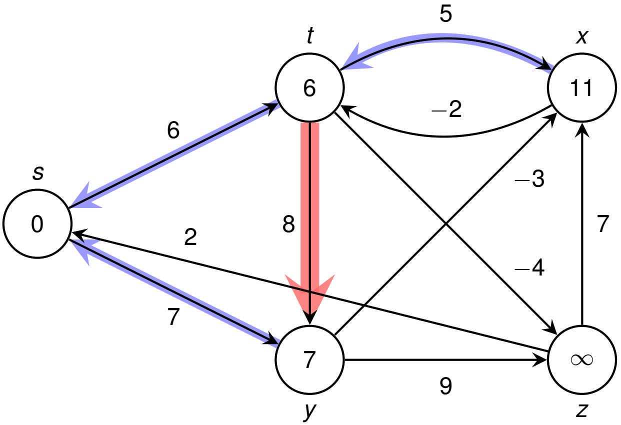

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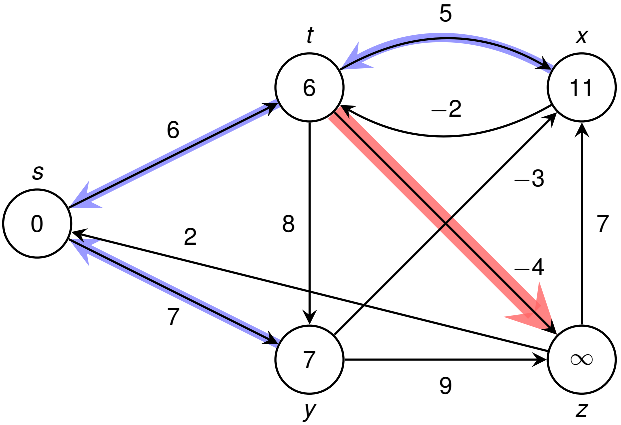

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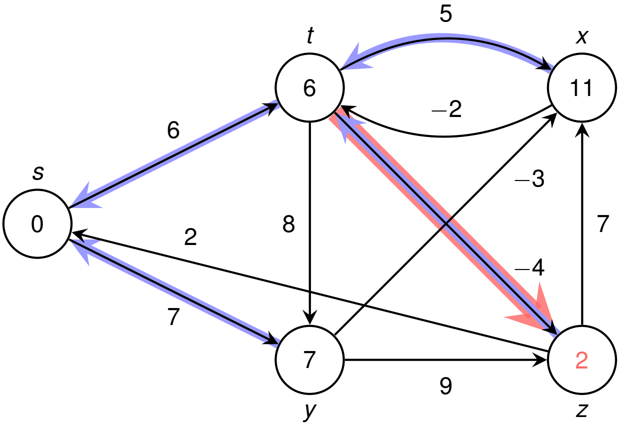
Pass: 2

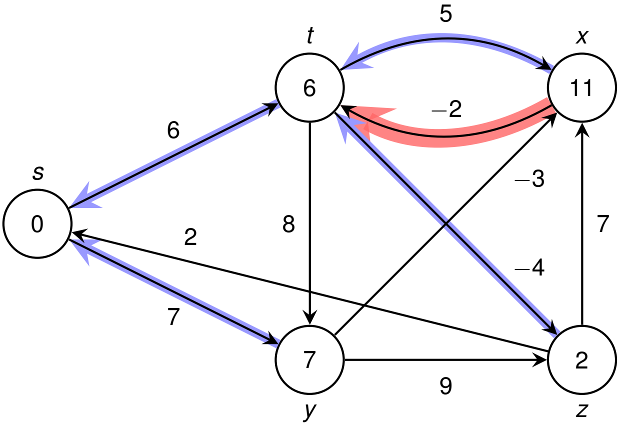
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 2

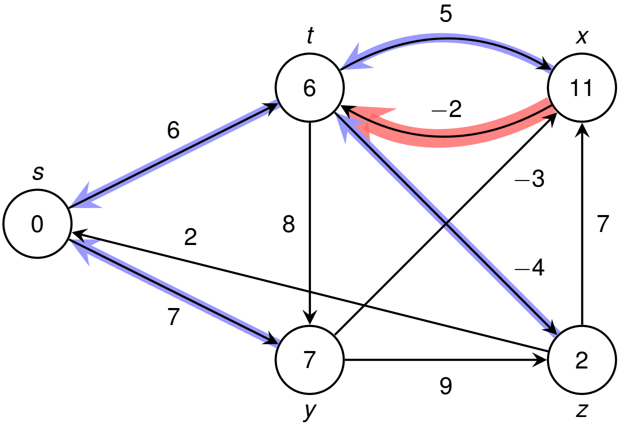
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 2

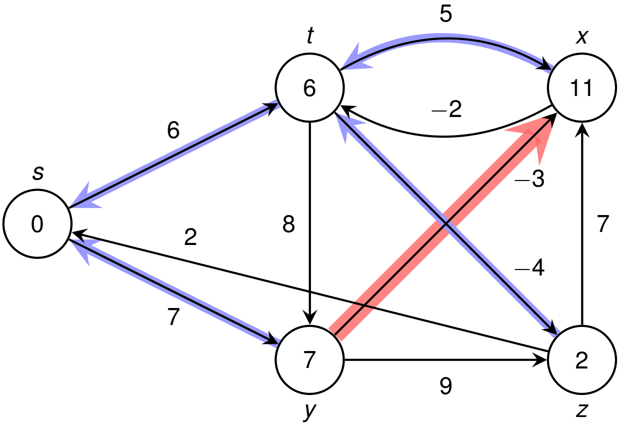
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 2

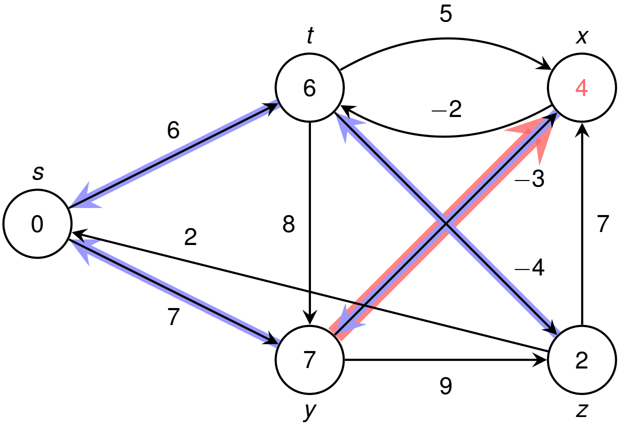
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 2

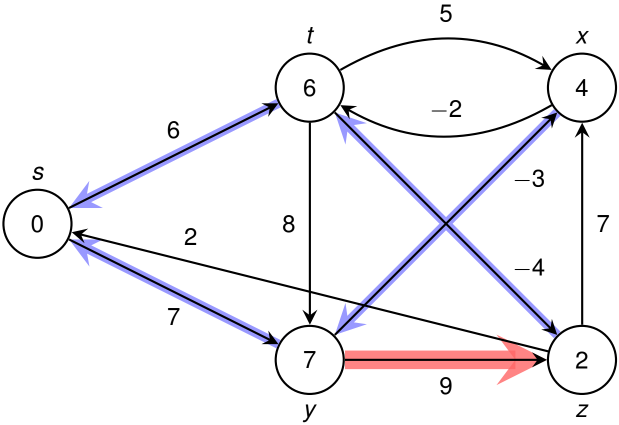
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 2

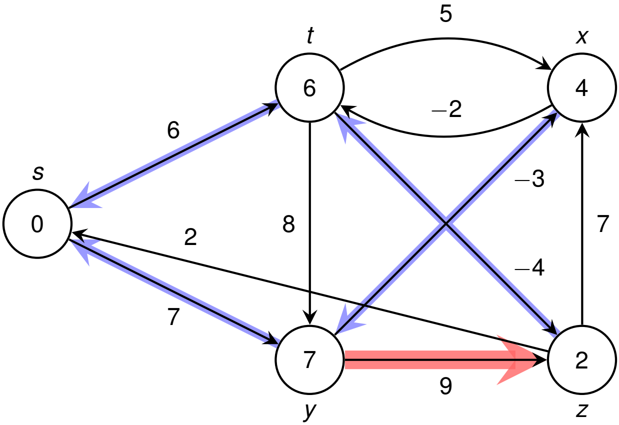
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 2

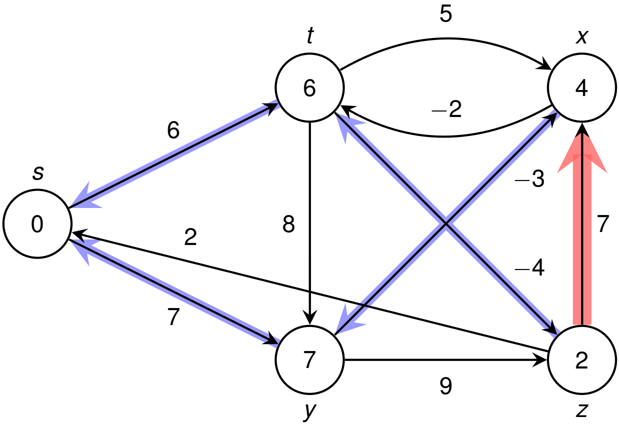
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 2

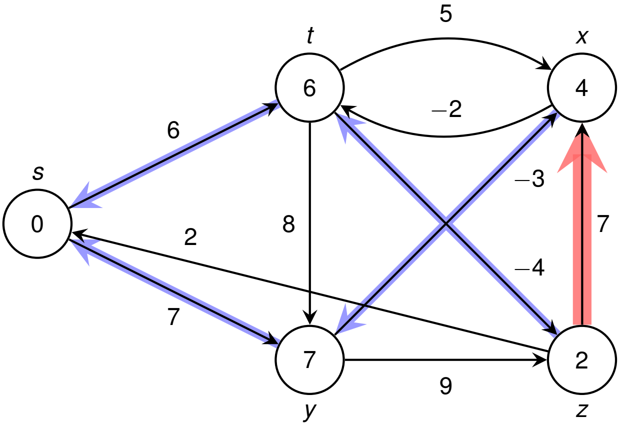
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 2

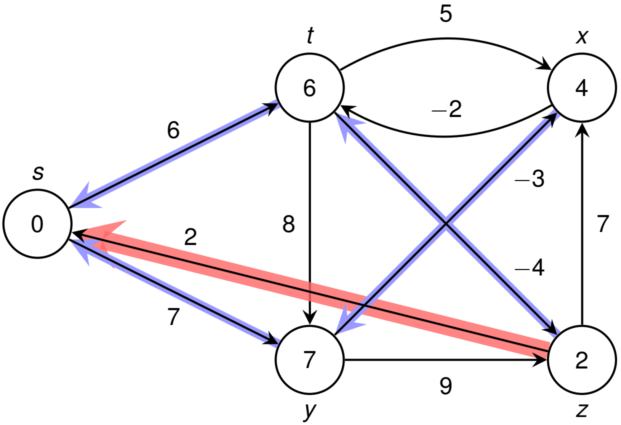
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 2

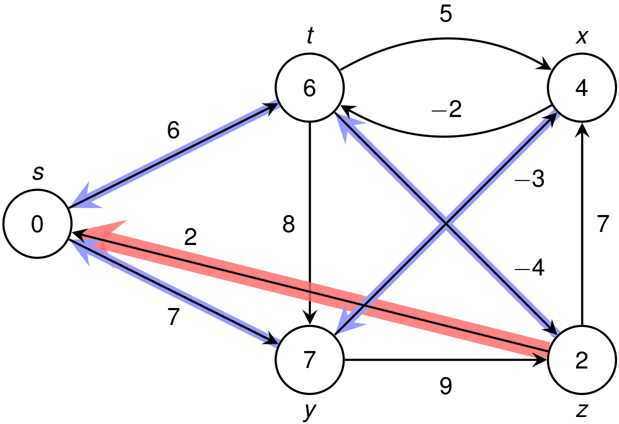
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 2

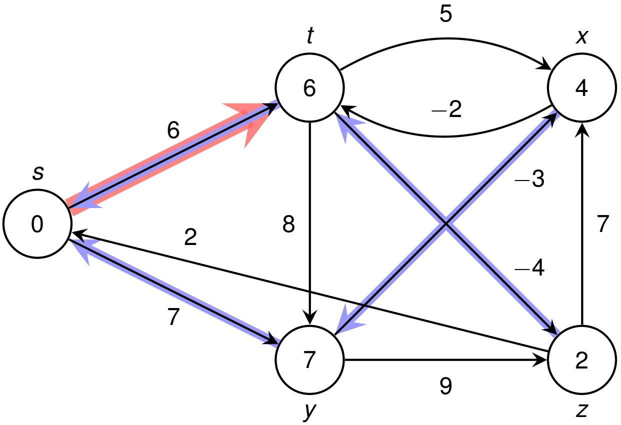
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 2

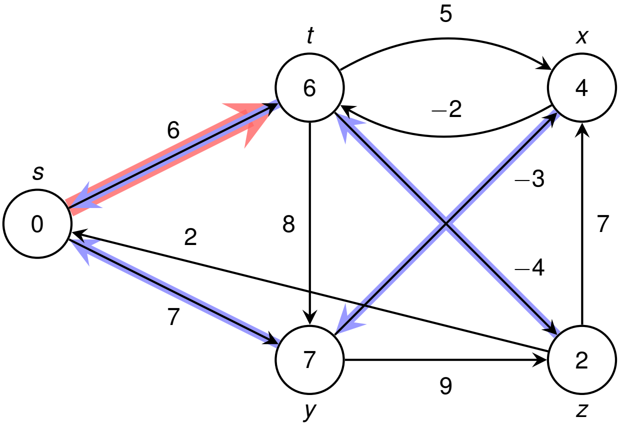
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 2

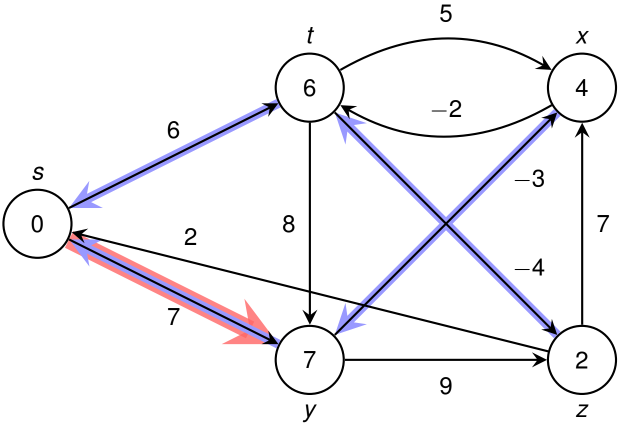
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 2

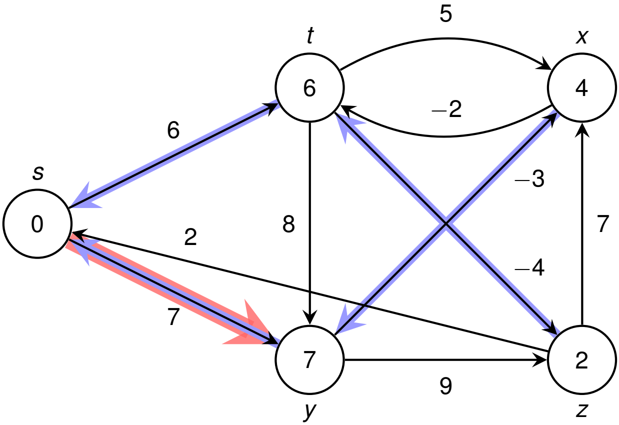
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 2

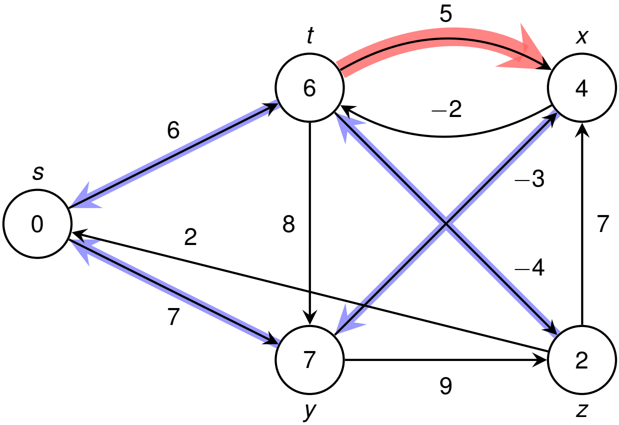
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

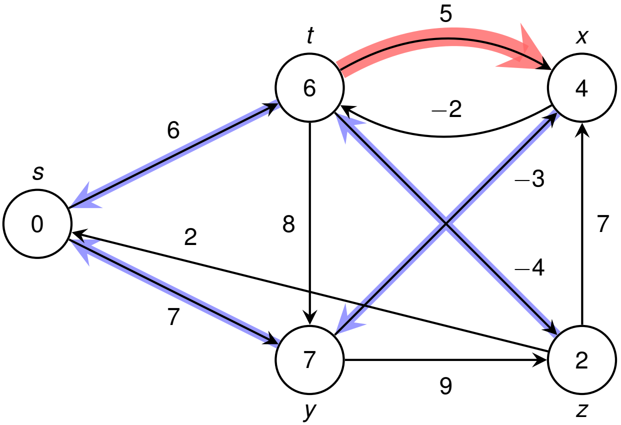
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

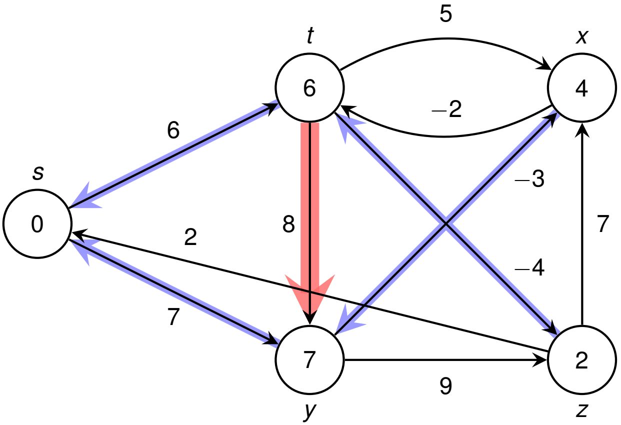
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

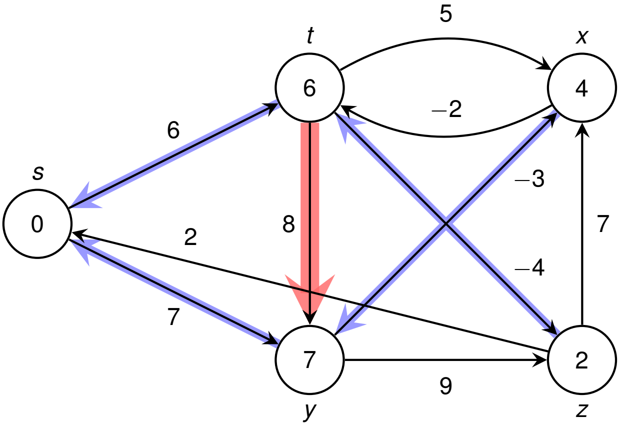
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

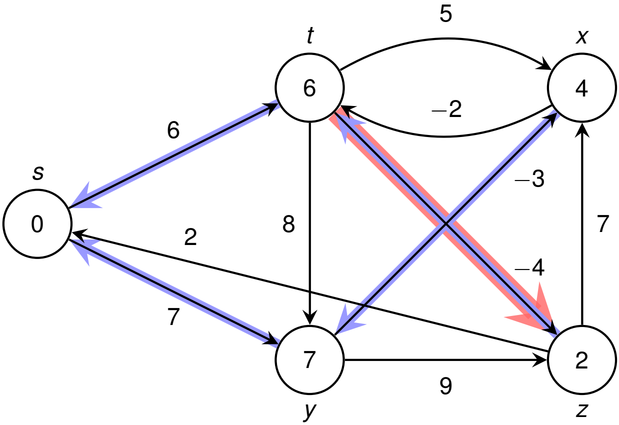
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

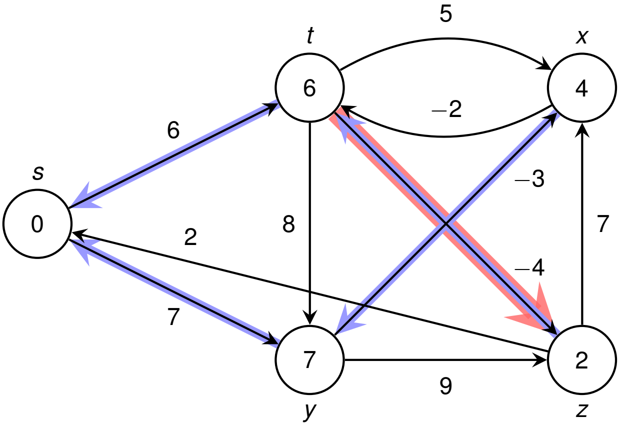
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

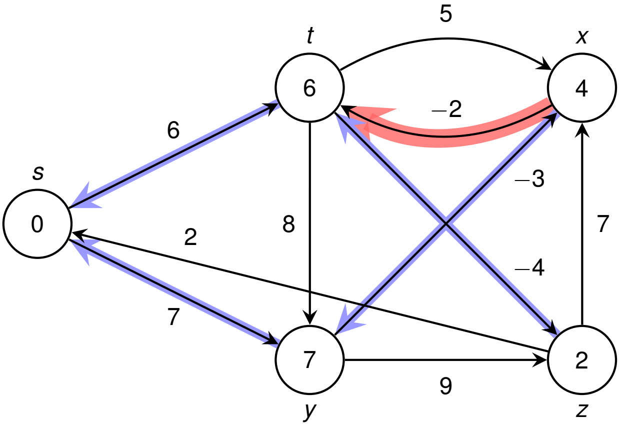
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

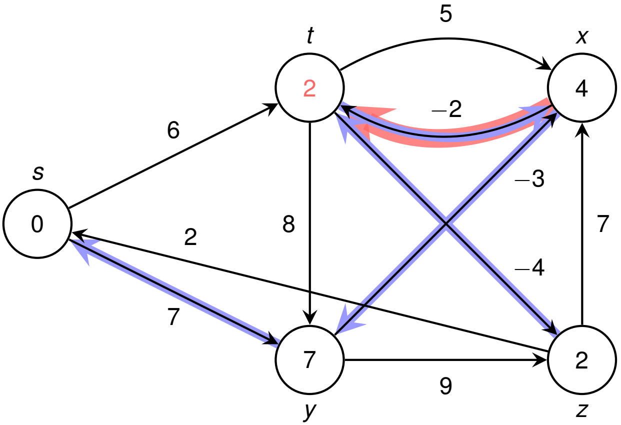
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

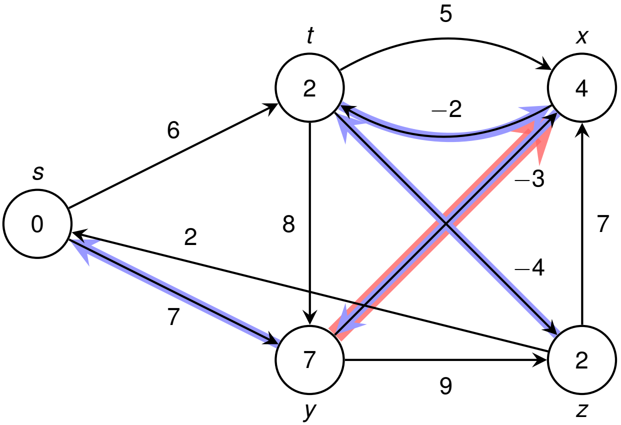
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

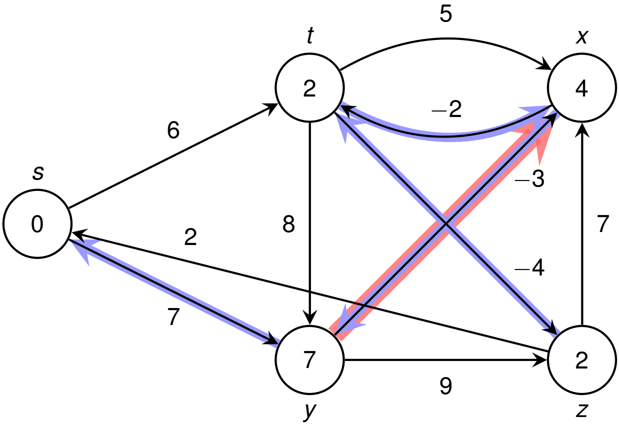
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

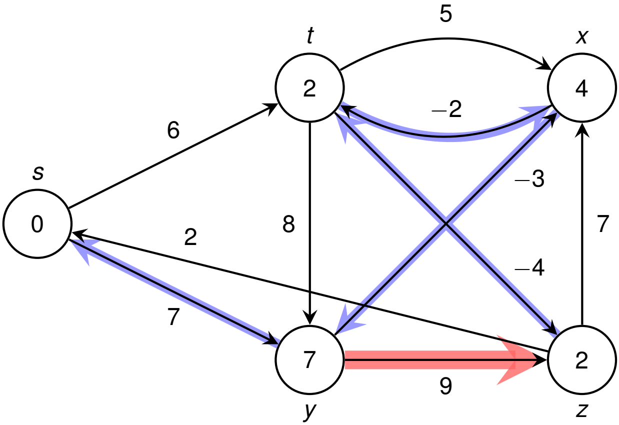
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

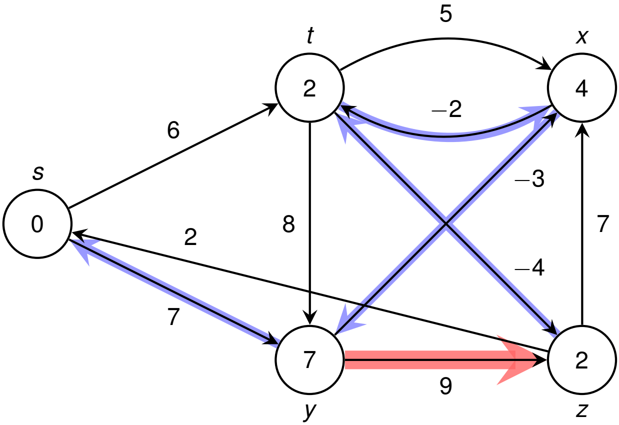
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

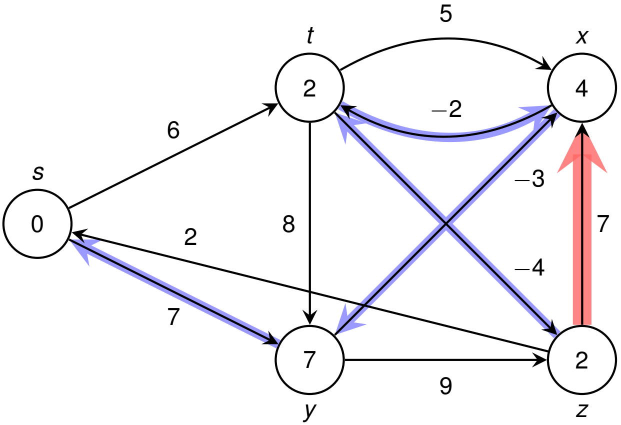
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

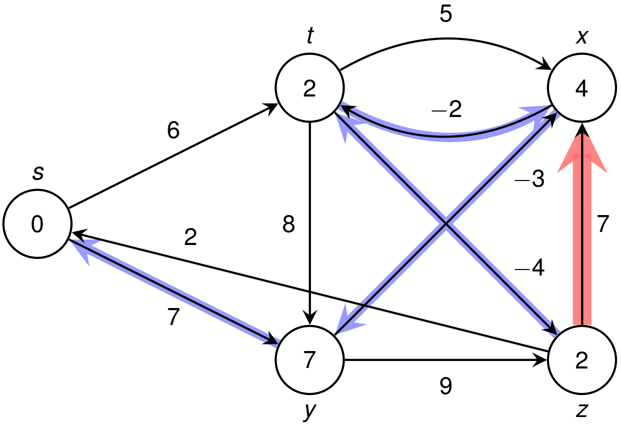
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

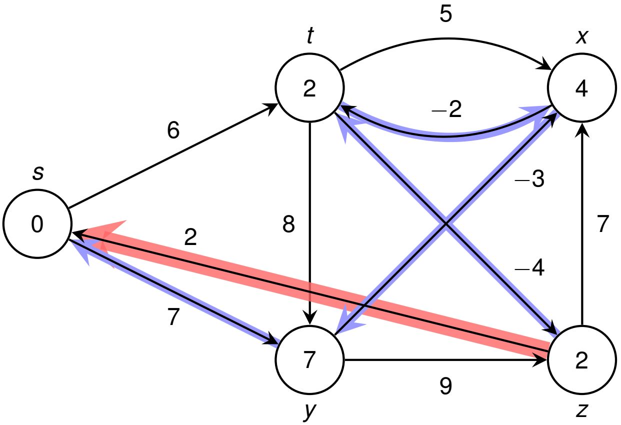
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

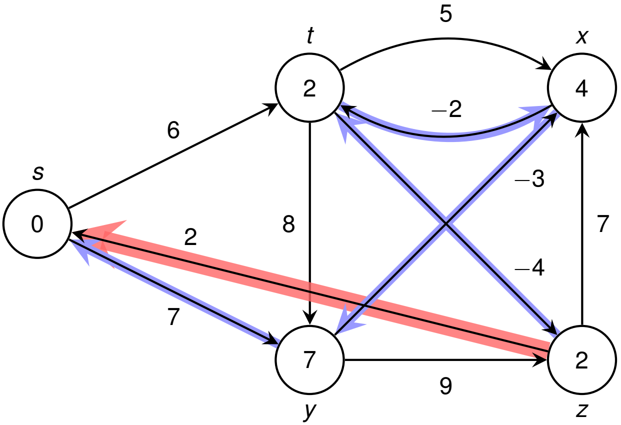
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

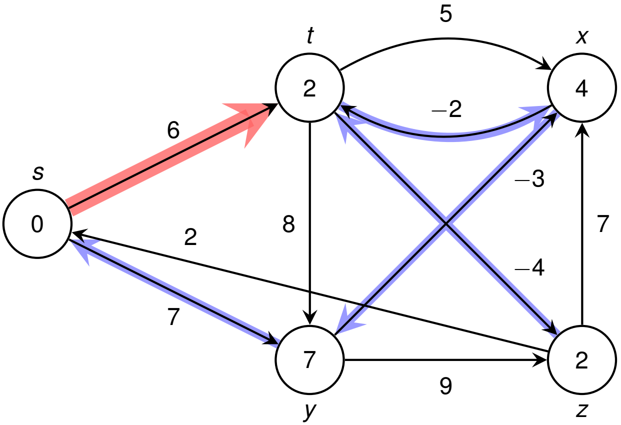
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

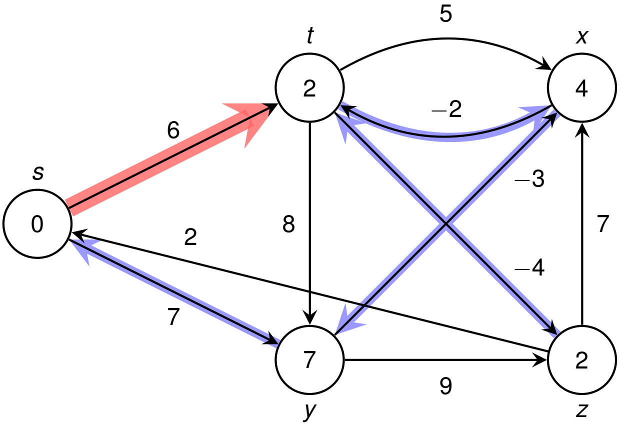
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

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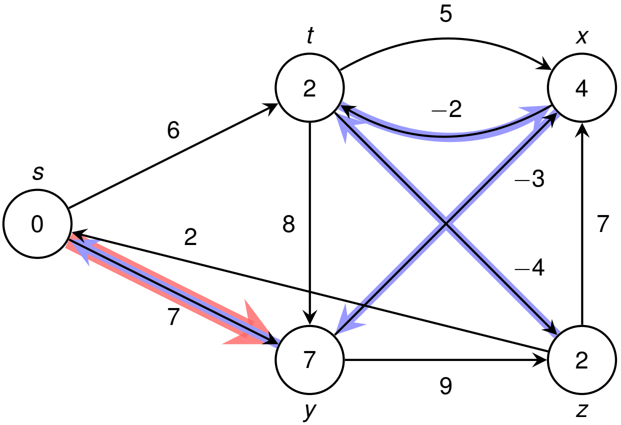
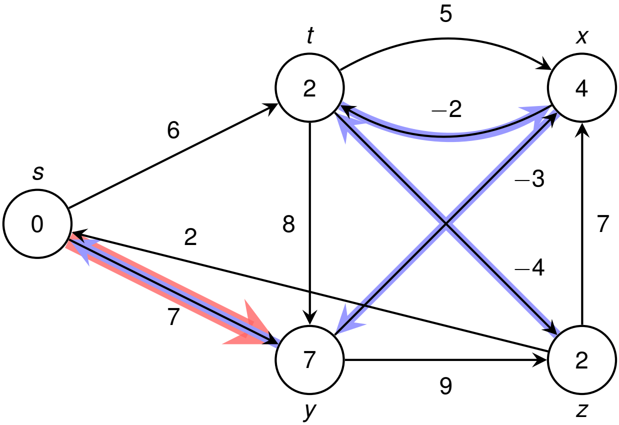
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 3

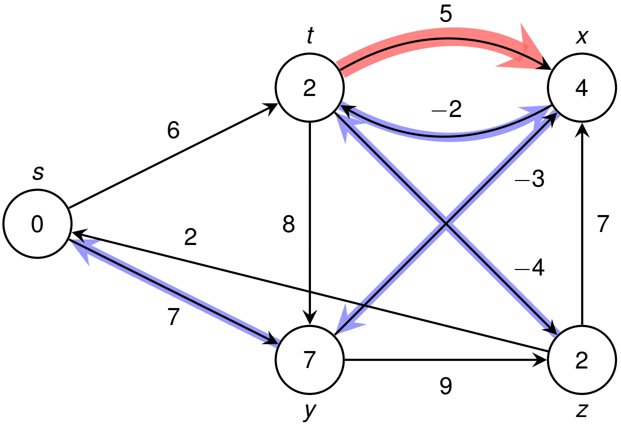
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 4

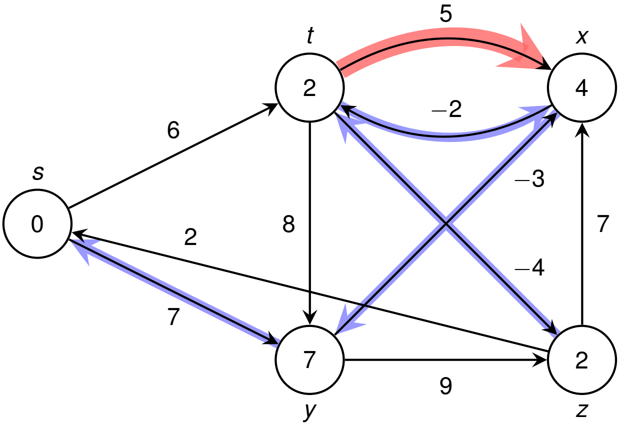
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 4

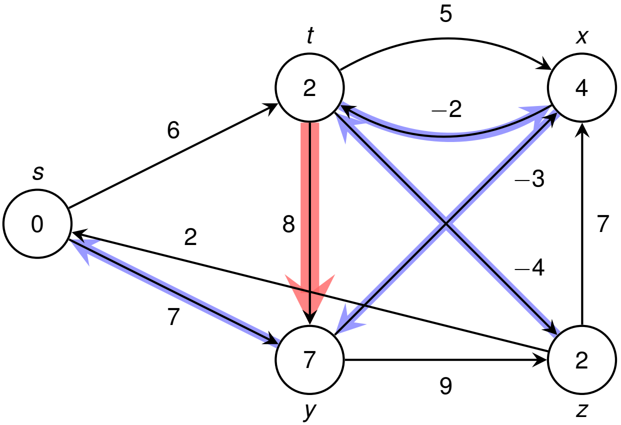
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 4

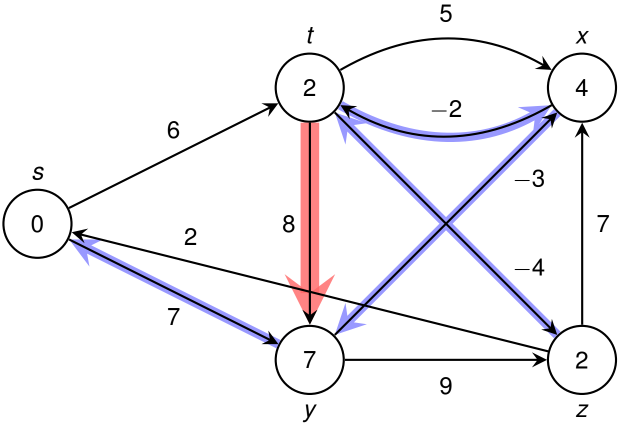
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 4

Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

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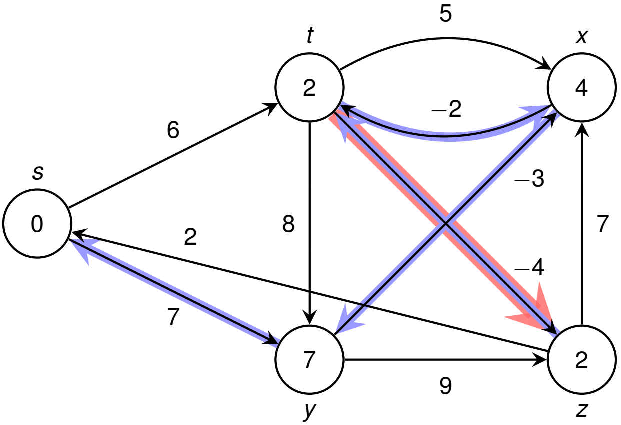
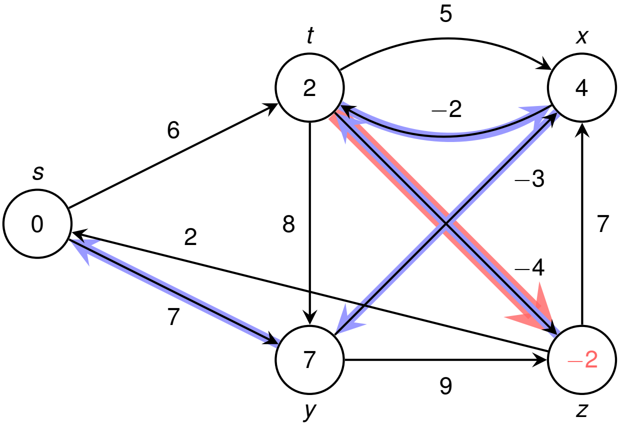
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 4

Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

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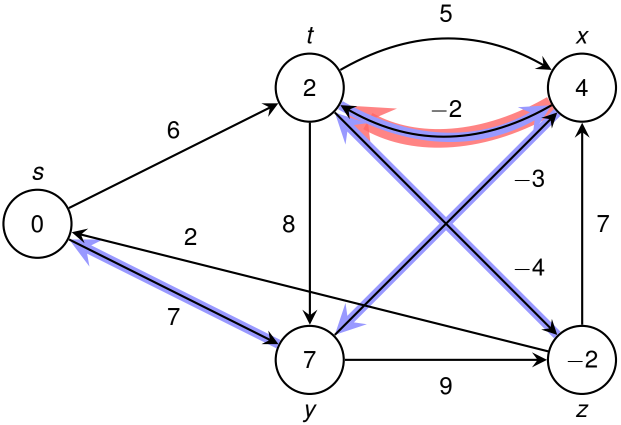
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

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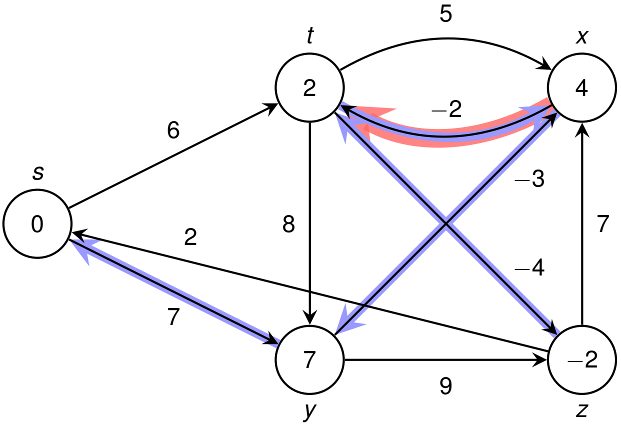
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

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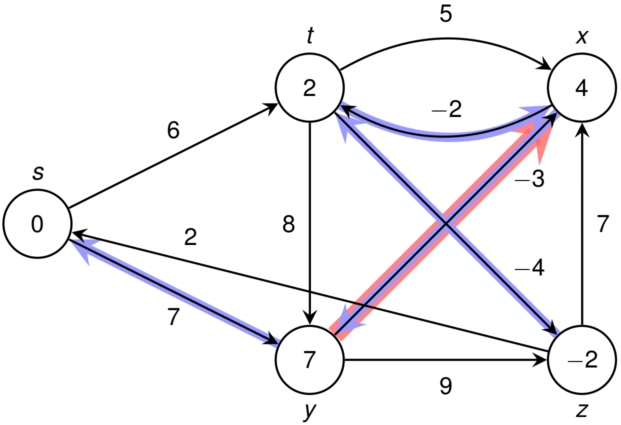
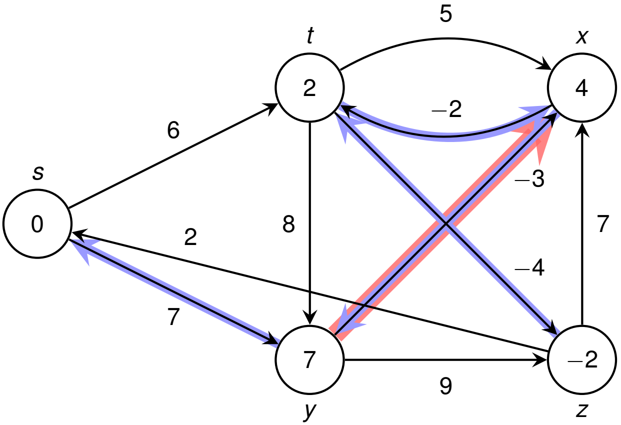
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 4

Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

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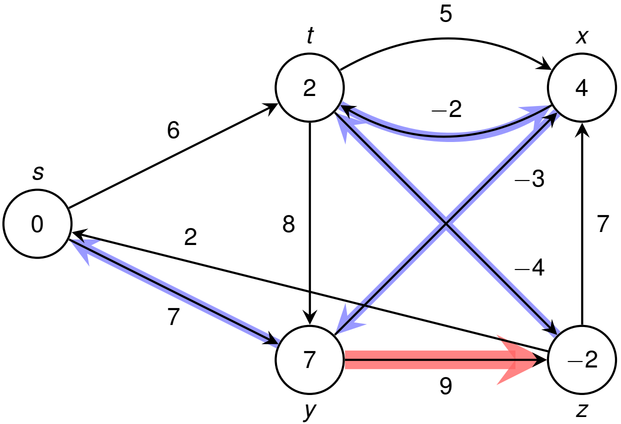
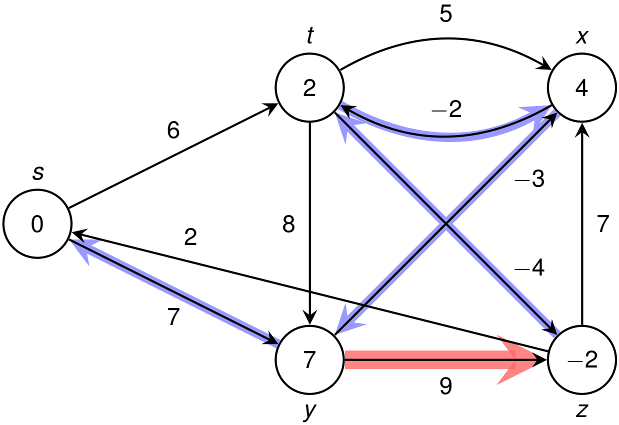
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 4

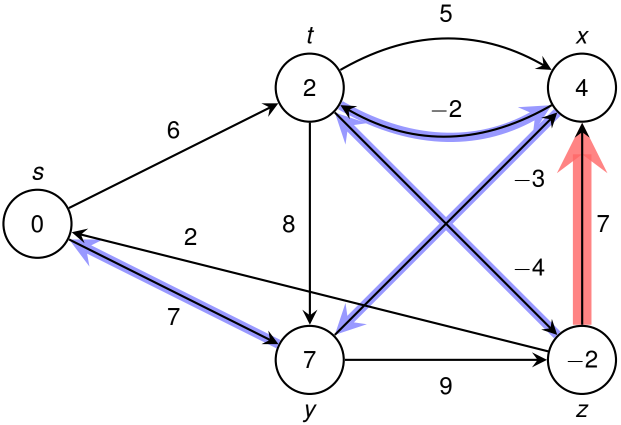
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 4

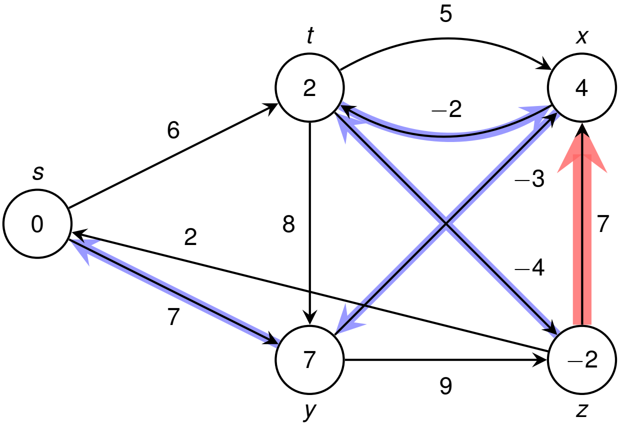
Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



Complete Run of Bellman-Ford (Figure 24.4)

Pass: 4

Relaxation Order: (t,x),(t,y),(t,z),(x,t),(y,x),(y,z),(z,x),(z,s),(s,t),(s,y)



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Pass: 4

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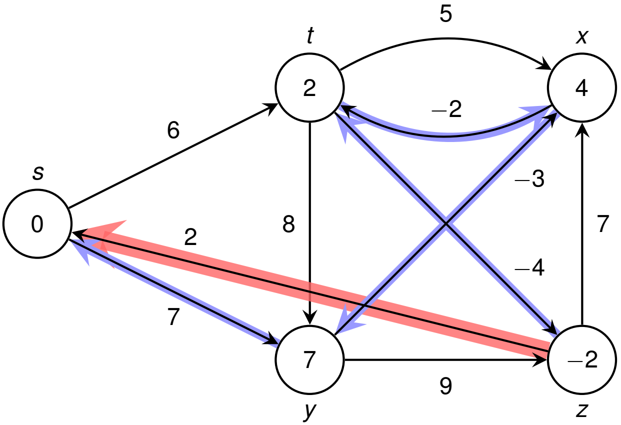

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Pass: 4

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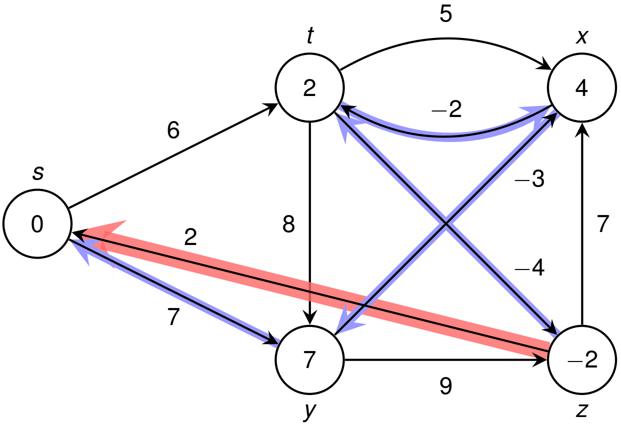

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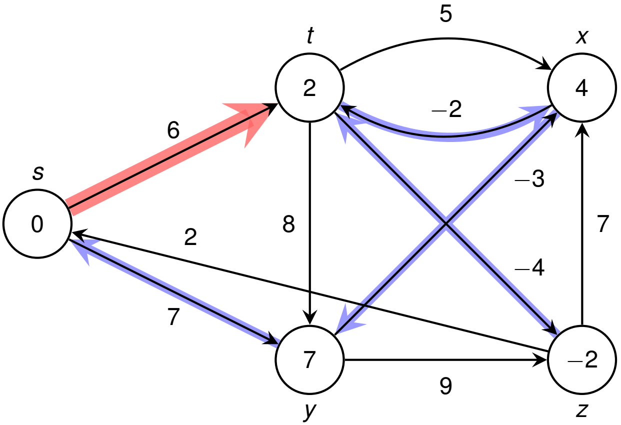

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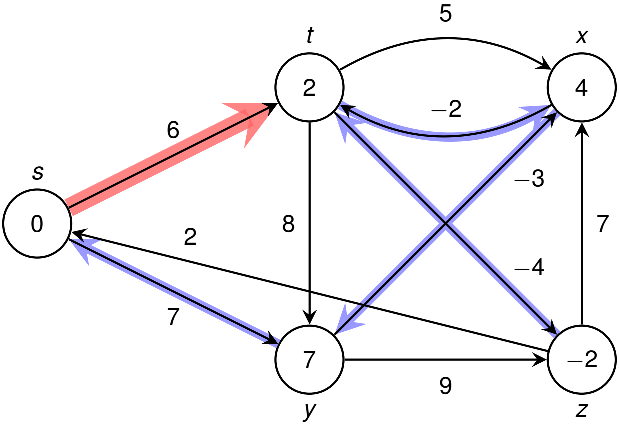

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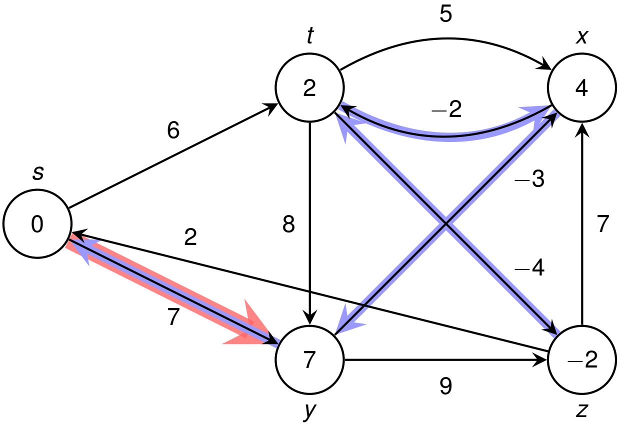

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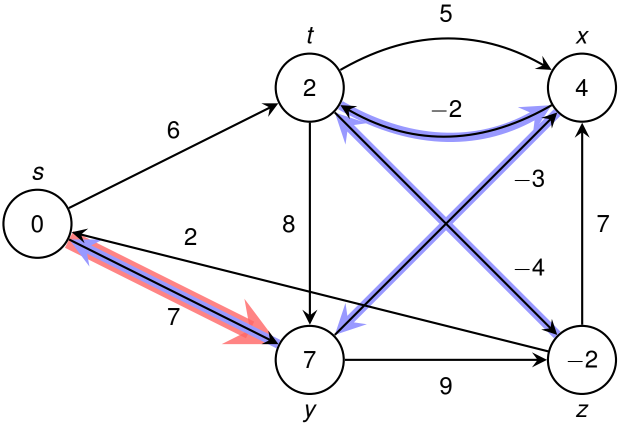

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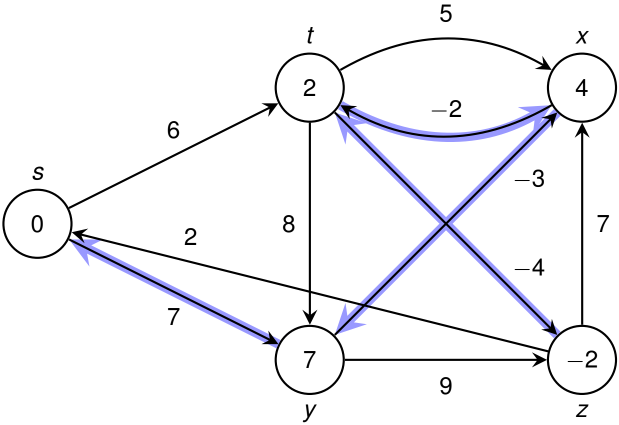

Bellman-Ford Algorithm: Correctness (1/2)

Lemma 24.2/Theorem 24.3

Assume that G contains no negative-weight cycles that are reachable from s. Then after $|V| - 1$ passes, we have $v.d = v.\delta$ for all vertices $v \in V$ that are reachable and Bellman-Ford returns TRUE.

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Proof that $v.d = v.\delta$

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Triangle inequality (holds even if $w(u,v) < 0$!)

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Theorem 24.3

If G contains a negative-weight cycle reachable from s, then Bellman-Ford returns FALSE.

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Proof:

- Let $c = (v_0, v_1, \ldots, v_k = v_0)$ be a negative-weight cycle reachable from s
- If Bellman-Ford returns TRUE, then for every $1 \leq i < k$,

$$v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$$

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This cancellation is only valid if all $.d$-values are finite!

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- This contradicts the assumption that c is a negative-weight cycle! $\square$