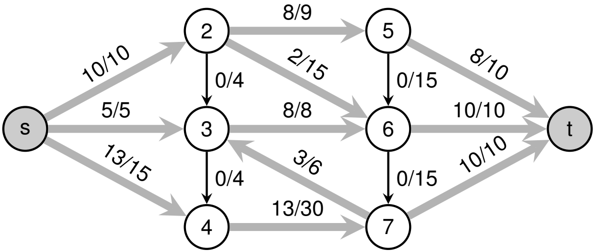
# Microchallenge 7

**Background:**

- Let $G$ be a flow network with integral capacities and without anti-parallel edges.
- An edge $e \in E$ is upper-binding if increasing its capacity by 1 also increases the maximum flow in $G$.
- An edge $e \in E$ is lower-binding if decreasing its capacity by 1 also decreases the maximum flow in $G$.

**Task:**

1. Develop an algorithm which, given $G$ and a maximum flow $f_{\max}$ in $G$ as input, computes all upper-binding edges in $G$ in time $O(E + V)$.
2. (★) Develop an algorithm which, given $G$ and a maximum flow $f_{\max}$ in $G$ as input, computes all lower-binding edges in $G$ as efficiently as possible.
3. Apply your algorithm(s) to the graph and maximum flow below by writing a small program.



## Upper-Binding Edges

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- Formally: Only increasing the capacity of such an edge leads to a new edge in the residual graph $G_f$ (and hence potentially to a new augmenting path from $s$ to $t$)

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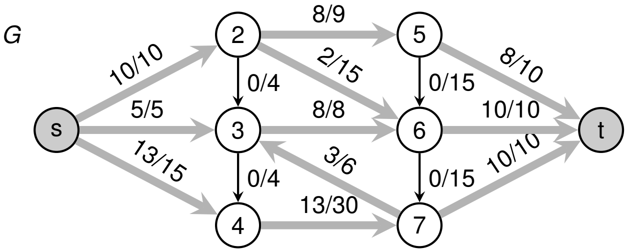

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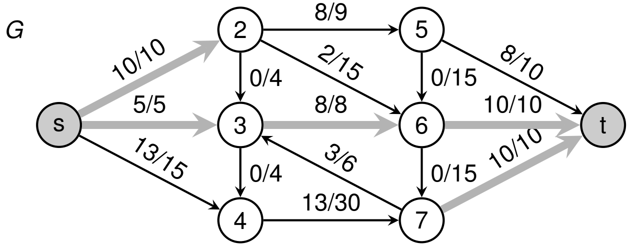

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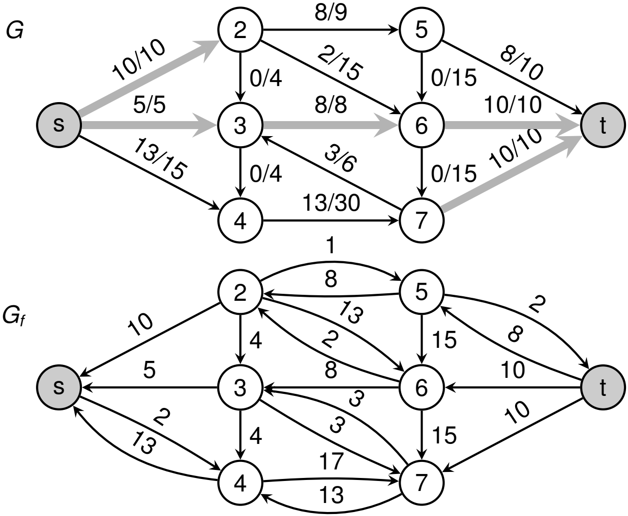

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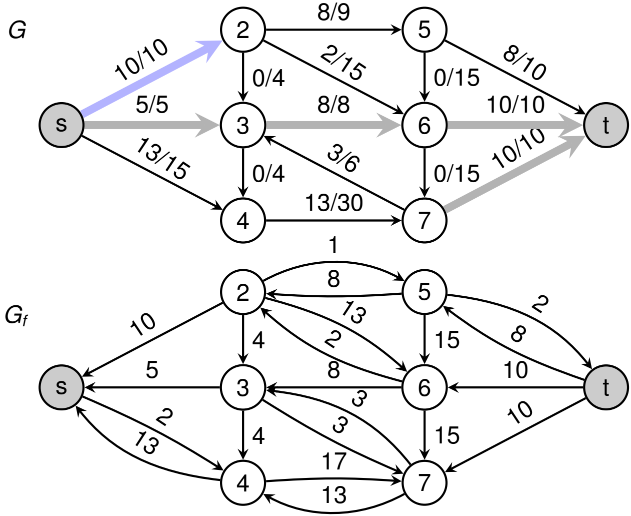

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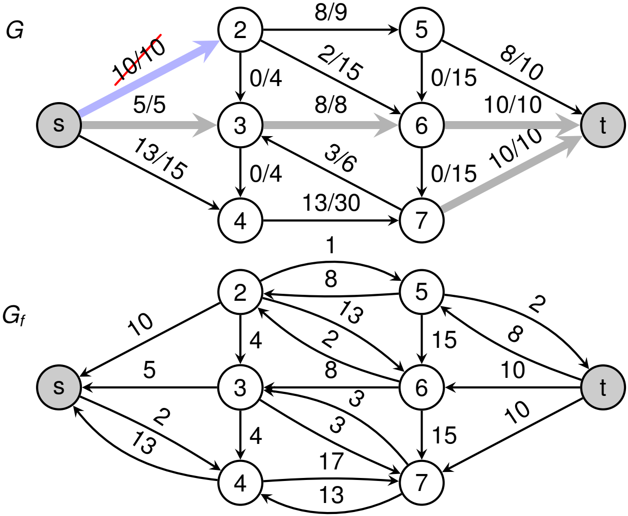

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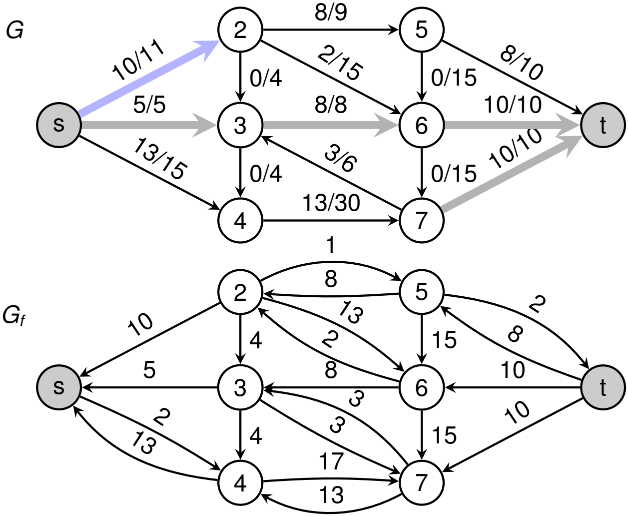

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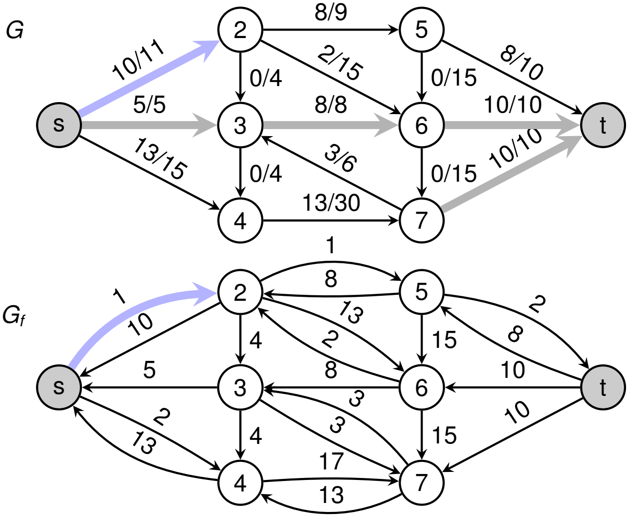

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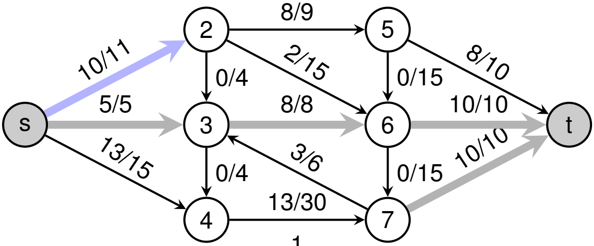
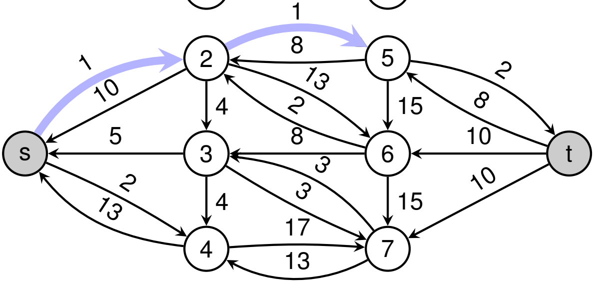
$G$




$G_f$



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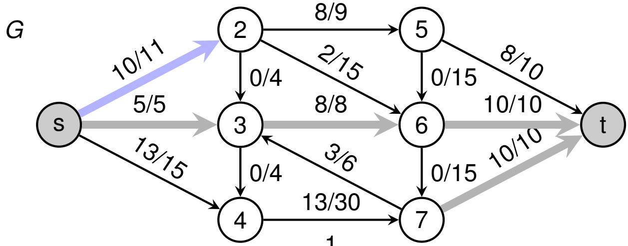


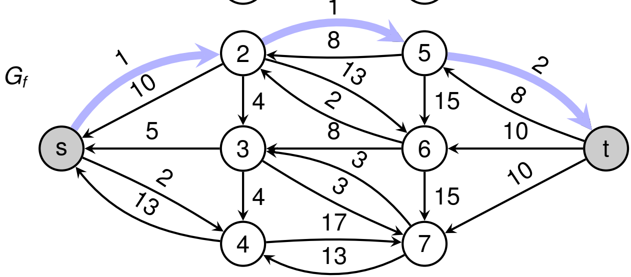

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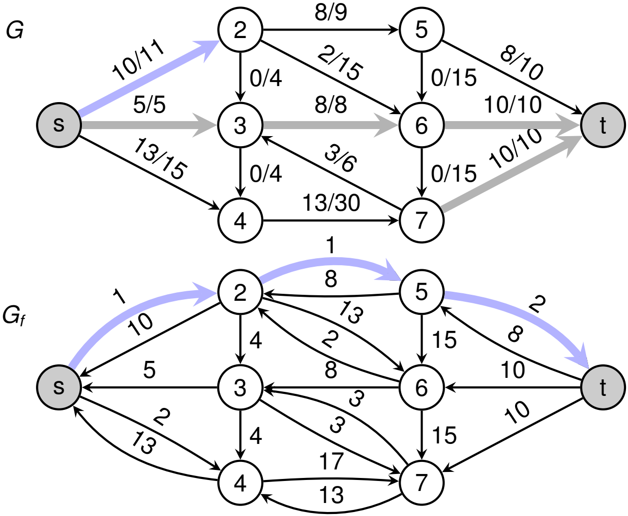


$(s, 2)$✓

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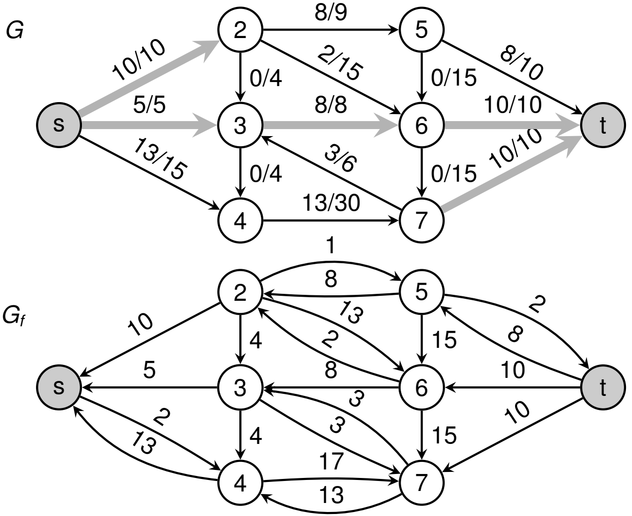


$(s, 2)\checkmark$

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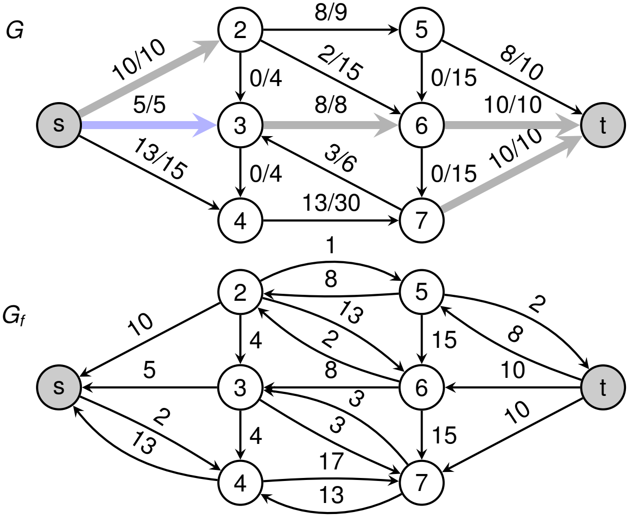


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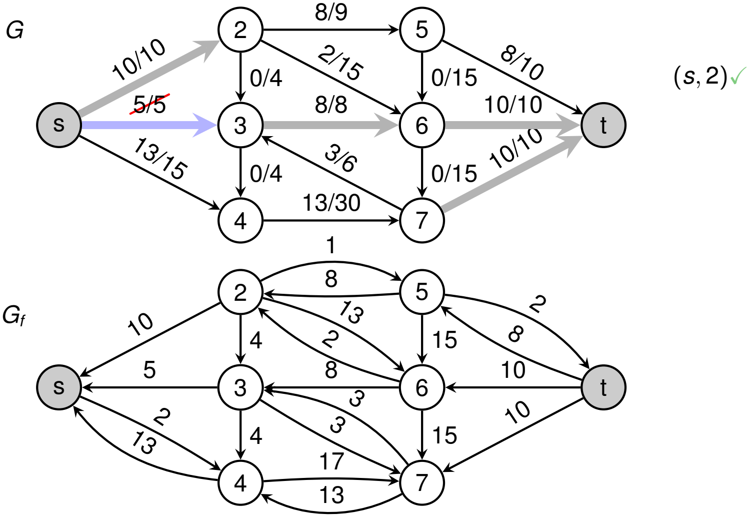

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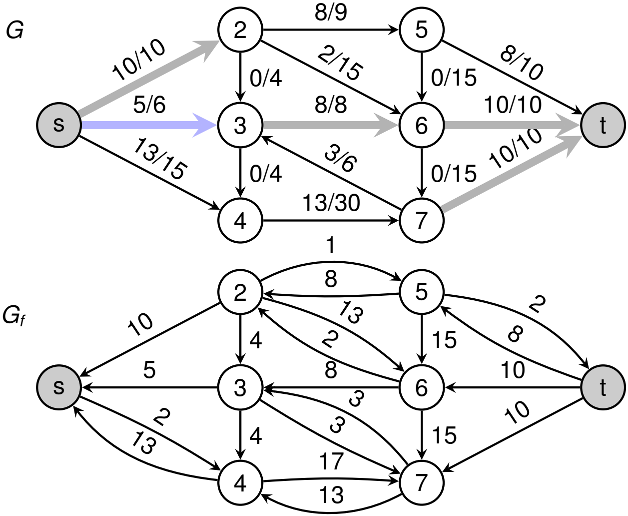


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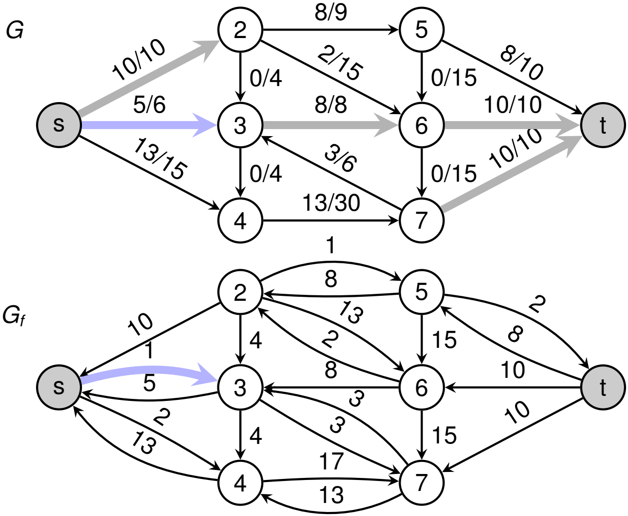


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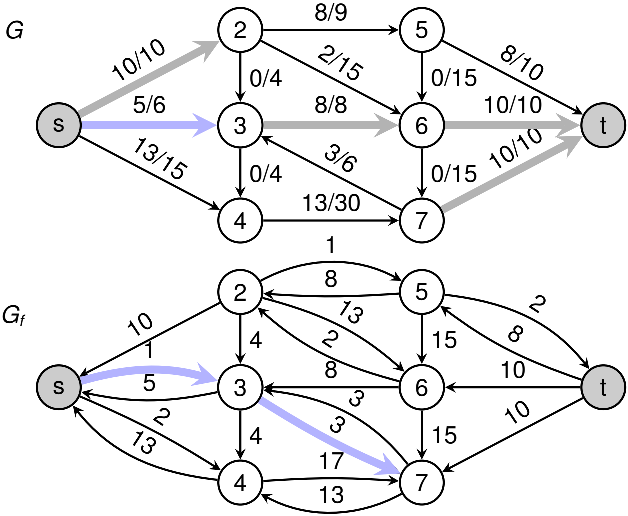


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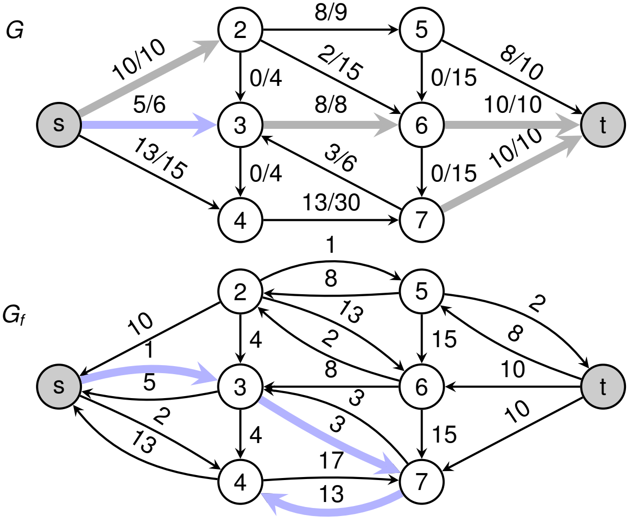


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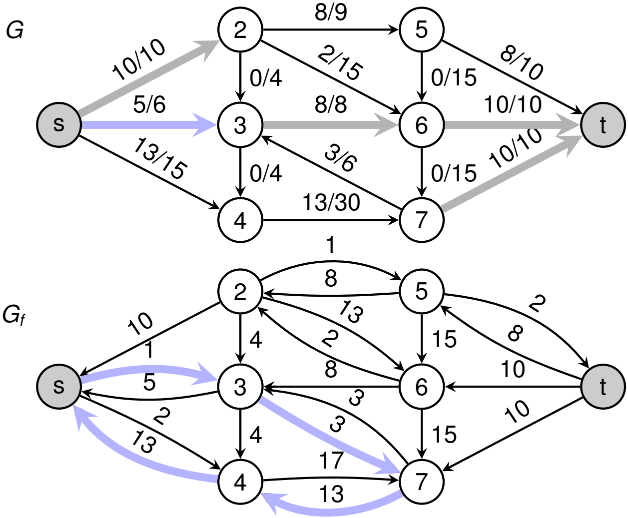


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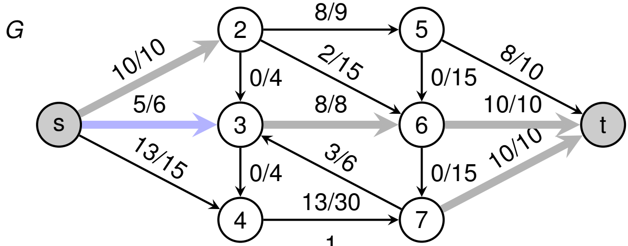


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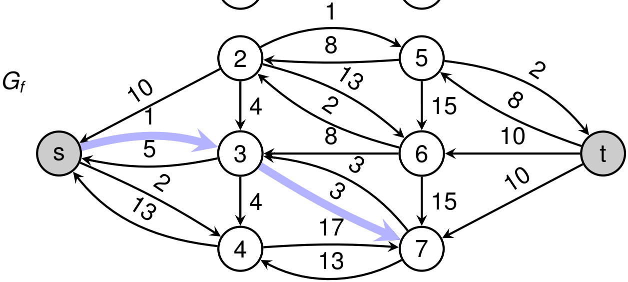

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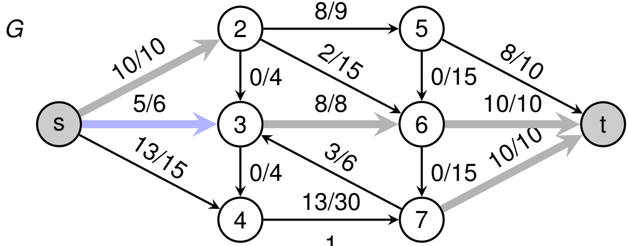


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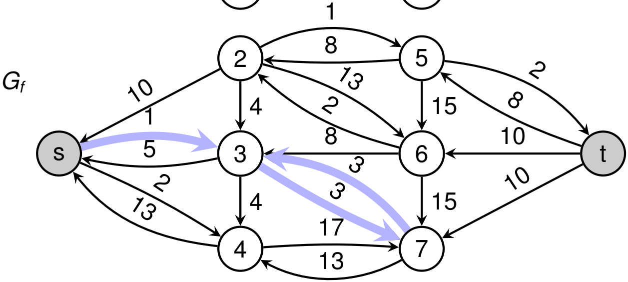

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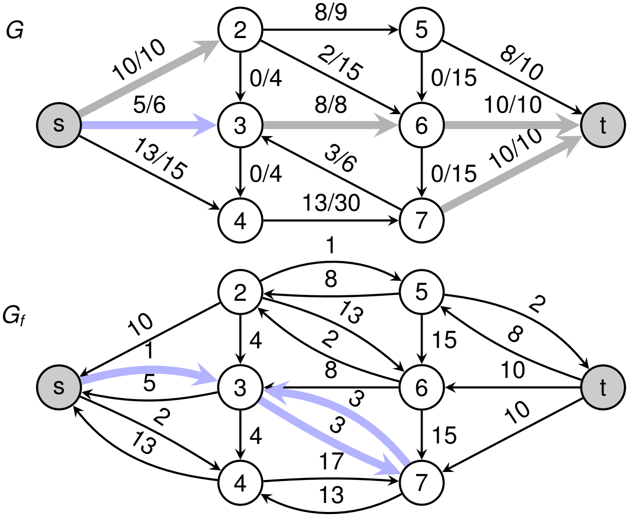


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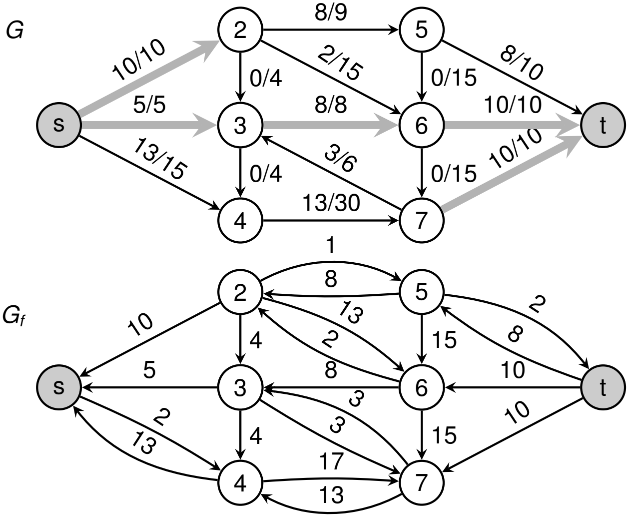


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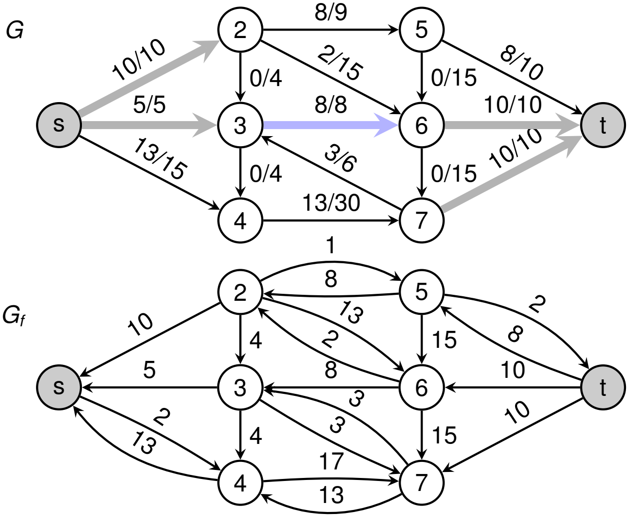


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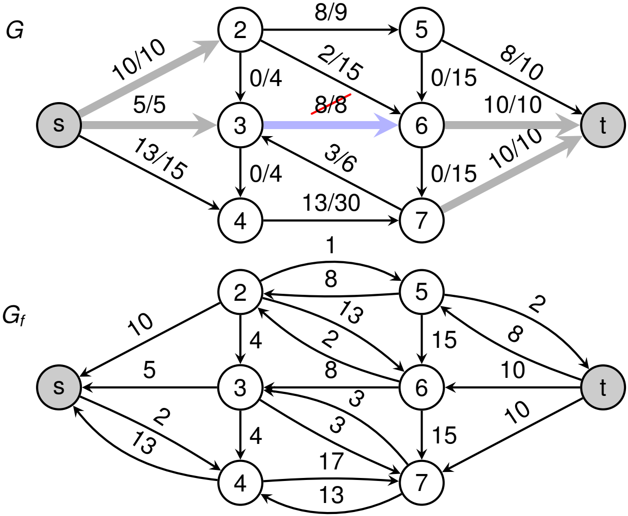


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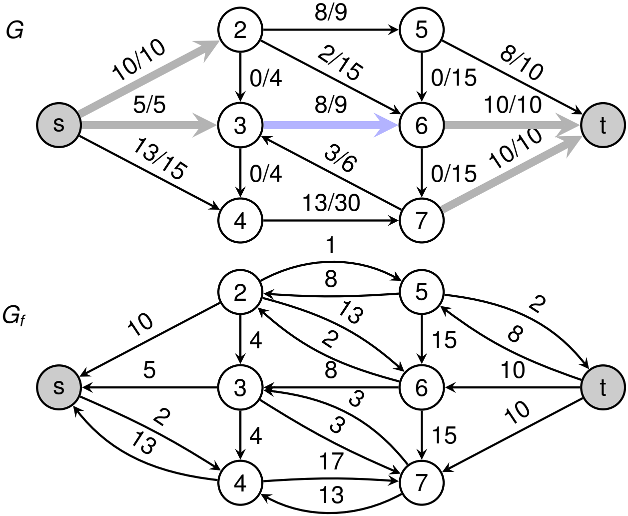


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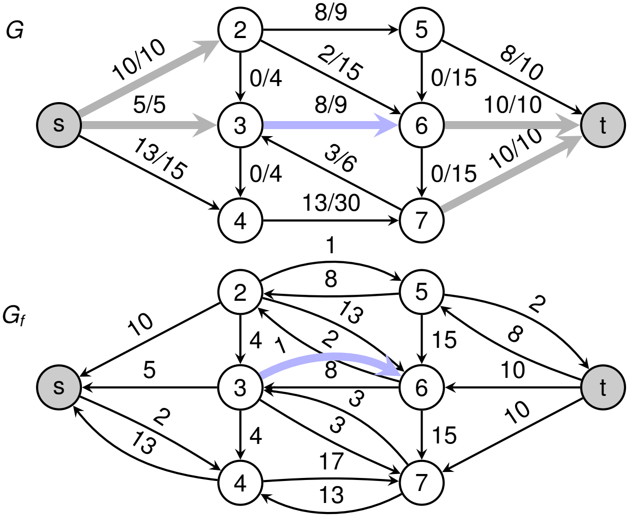


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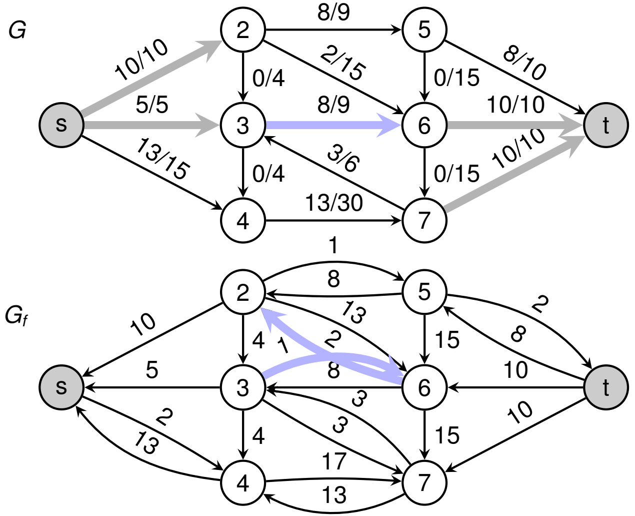


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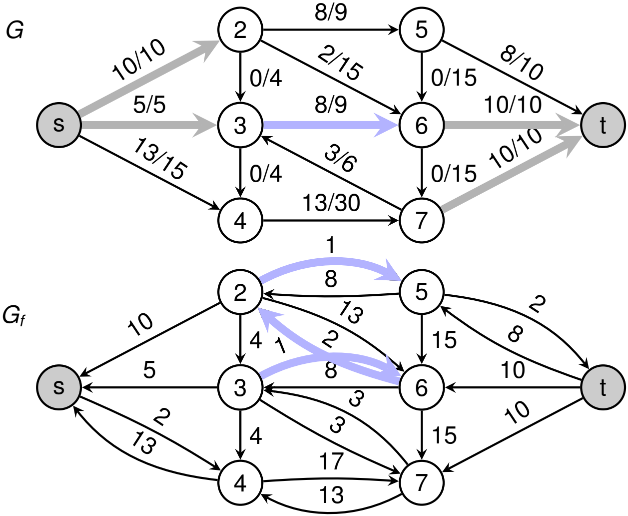


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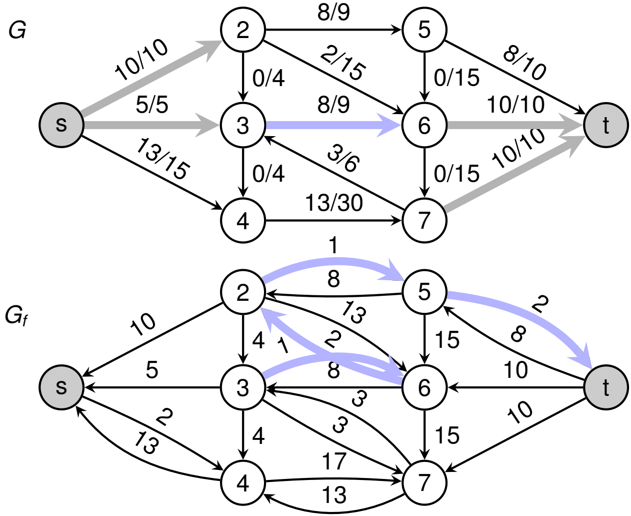


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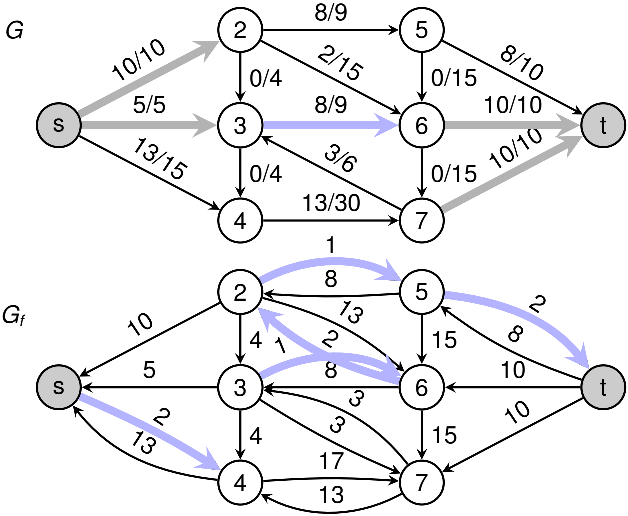


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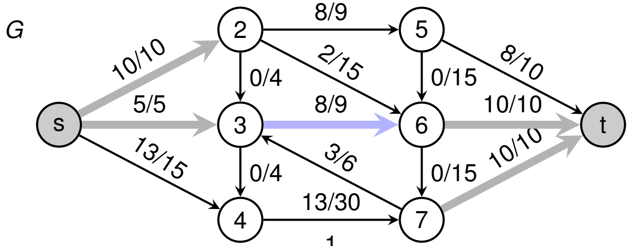


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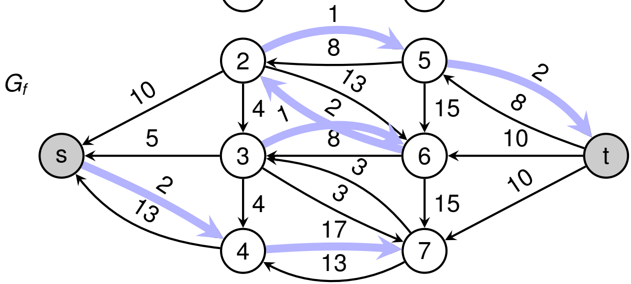
$\cancel{(s,3)}$



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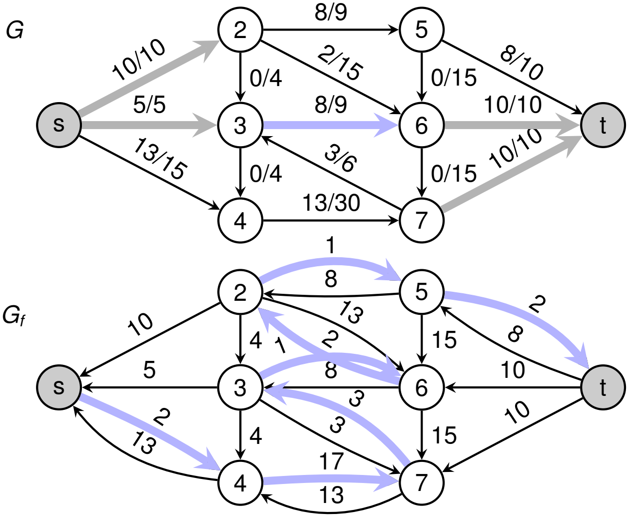


$(s, 2)$ ✓

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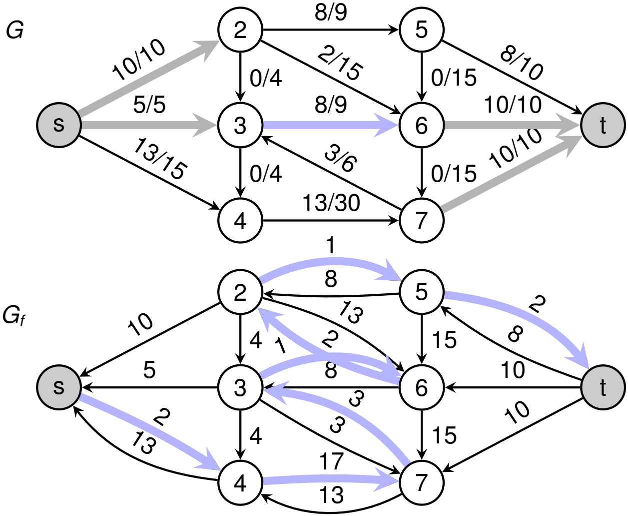


$(s,2)$✓

$\cancel{(s,3)}$

$(3,6)$✓

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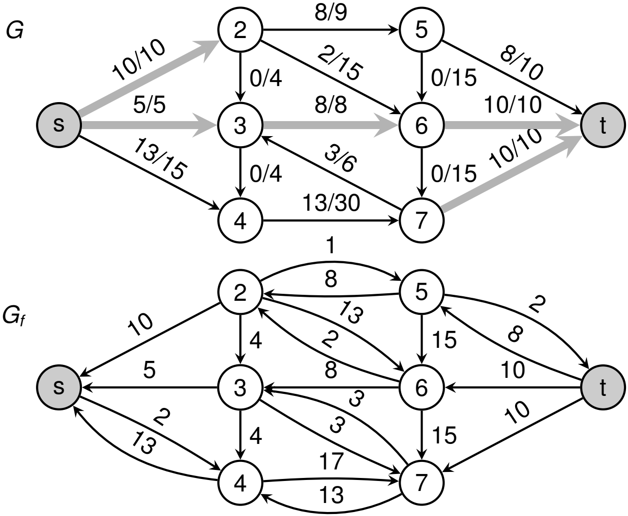


$(s,2)$ ✓

~~$(s,3)$~~

$(3,6)$ ✓

~~$(6,t)$~~

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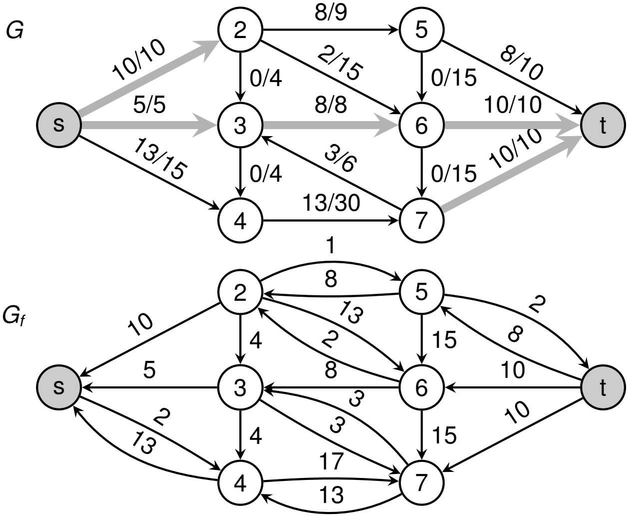


$(s,2)$ ✓

~~$(s,3)$~~

$(3,6)$ ✓

~~$(6,t)$~~

$(7,t)$ ✓

## Upper-Binding Edges

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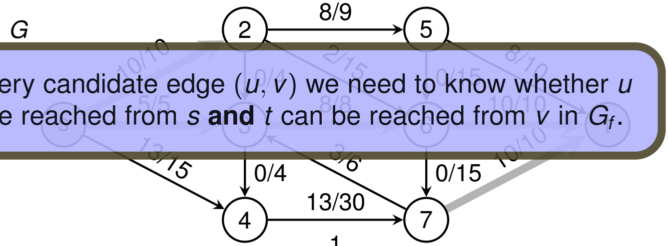



For every candidate edge $(u, v)$ we need to know whether $u$ can be reached from $s$ **and** $t$ can be reached from $v$ in $G_f$.

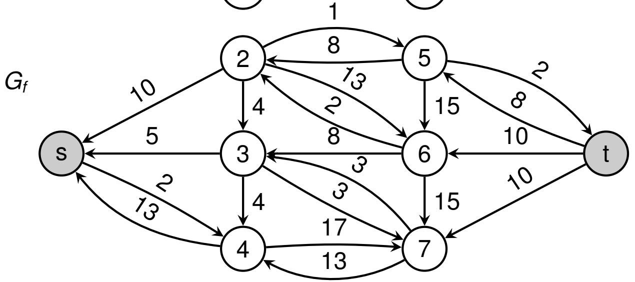
$(s, 2)$ ✓

$\cancel{(s, 3)}$

$(3, 6)$ ✓

$\cancel{(6, t)}$

$(7, t)$ ✓



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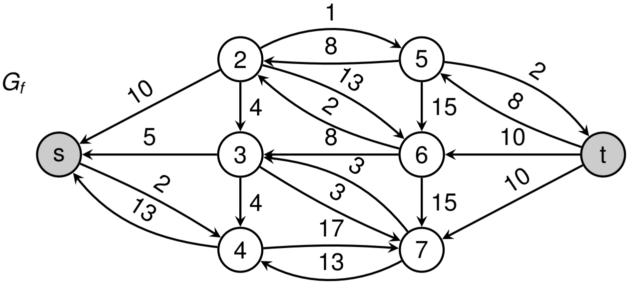
$(3, 6)$✓

$\cancel{(6, t)}$

$(7, t)$✓



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⇒ only edges which are used to their capacity can be upper-binding

Algorithm:

$(s, 2)$ ✓

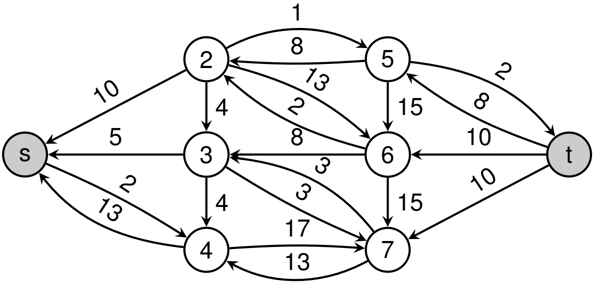
~~$(s, 3)$~~

$(3, 6)$ ✓

~~$(6, t)$~~

$(7, t)$ ✓

$G_f$



## Upper-Binding Edges

$\Rightarrow$ only edges which are used to their capacity can be upper-binding

Algorithm:

1. Let $S$ be all vertices reachable from $s$

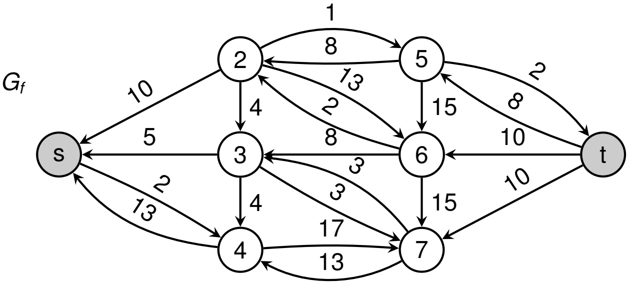
$(s,2)$ ✓

~~$(s,3)$~~

$(3,6)$ ✓

~~$(6,t)$~~

$(7,t)$ ✓



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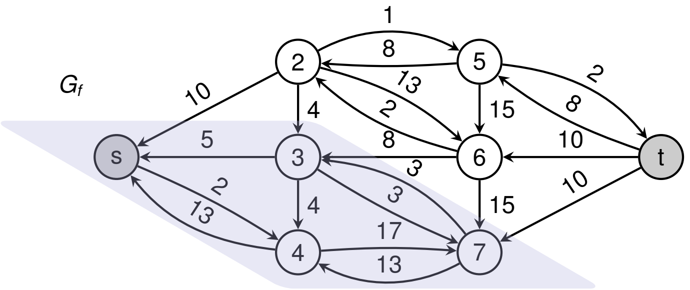
$(7,t)$ ✓



## Upper-Binding Edges

$\Rightarrow$ only edges which are used to their capacity can be upper-binding

Algorithm:

1. Let $S$ be all vertices reachable from $s$ (DFS on $G_f$ from $s$)

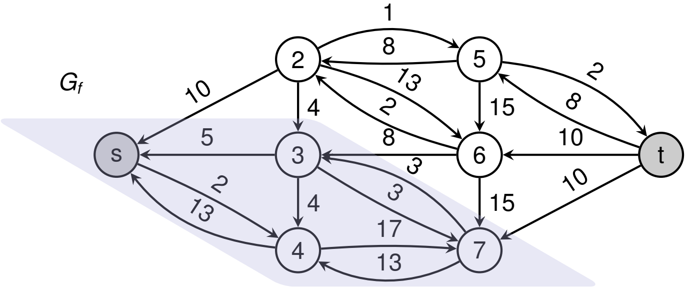
$(s, 2)$✓

$\cancel{(s, 3)}$

$(3, 6)$✓

$\cancel{(6, t)}$

$(7, t)$✓



# Upper-Binding Edges

⇒ only edges which are used to their capacity can be upper-binding

Algorithm:

1. Let $S$ be all vertices reachable from $s$ (DFS on $G_f$ from $s$)
2. Let $T$ be all vertices that can reach $t$

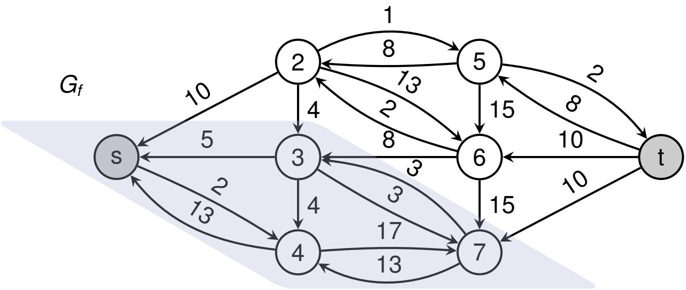
$(s, 2)$ ✓

~~$(s, 3)$~~

$(3, 6)$ ✓

~~$(6, t)$~~

$(7, t)$ ✓



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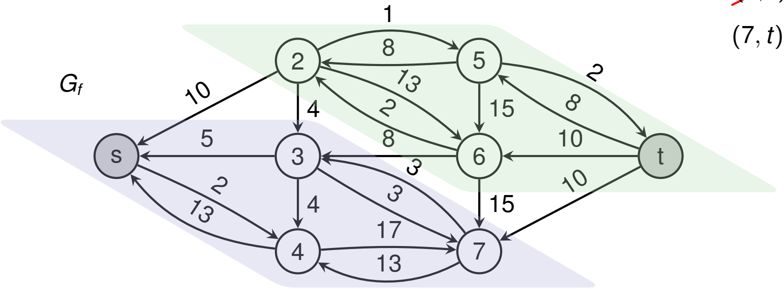
$(s, 2)$✓

$\cancel{(s, 3)}$

$(3, 6)$✓

$\cancel{(6, t)}$

$(7, t)$✓



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Algorithm:

1. Let $S$ be all vertices reachable from $s$ (DFS on $G_f$ from $s$)
2. Let $T$ be all vertices that can reach $t$ (DFS on reversed $G_f$ from $t$)

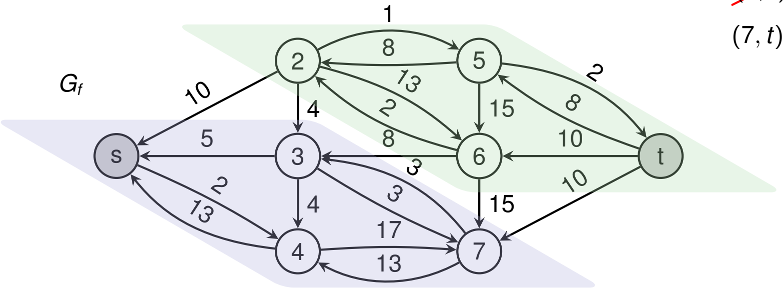
$(s, 2)$✓

~~$(s, 3)$~~

$(3, 6)$✓

~~$(6, t)$~~

$(7, t)$✓



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Algorithm:

1. Let $S$ be all vertices reachable from $s$ (DFS on $G_f$ from $s$)
2. Let $T$ be all vertices that can reach $t$ (DFS on reversed $G_f$ from $t$)
3. Go through all candidates $(u, v)$ and check if $u \in S$ and $v \in T$

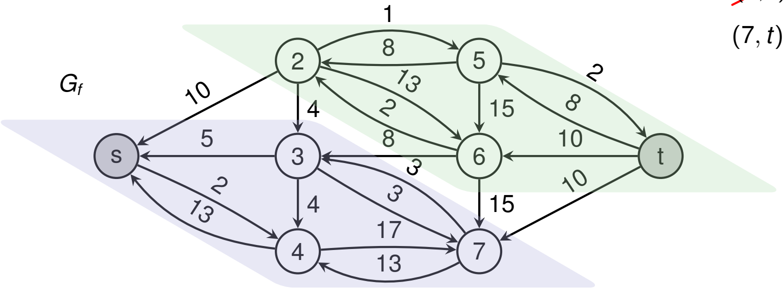
$(s, 2)$ ✓

$\cancel{(s, 3)}$

$(3, 6)$ ✓

$\cancel{(6, t)}$

$(7, t)$ ✓



# Upper-Binding Edges

$\Rightarrow$ only edges which are used to their capacity can be upper-binding

Algorithm:

1. Let $S$ be all vertices reachable from $s$ (DFS on $G_f$ from $s$)
2. Let $T$ be all vertices that can reach $t$ (DFS on reversed $G_f$ from $t$)
3. Go through all candidates $(u, v)$ and check if $u \in S$ and $v \in T$

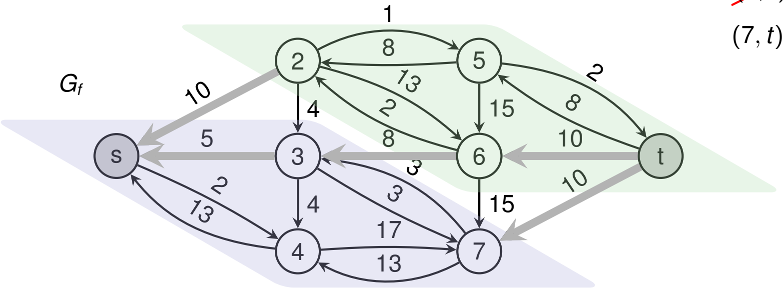
$(s, 2)$ ✓

$\cancel{(s, 3)}$

$(3, 6)$ ✓

$\cancel{(6, t)}$

$(7, t)$ ✓



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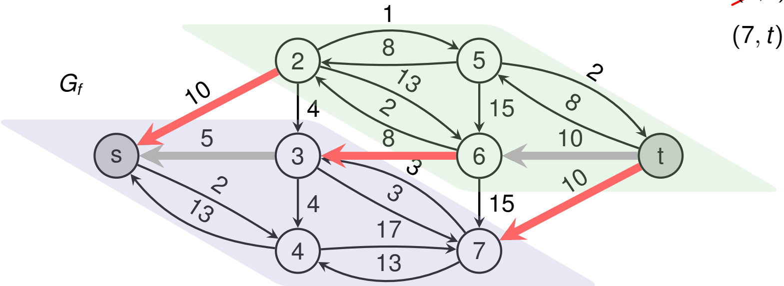
$(s, 2)$✓

~~$(s, 3)$~~

$(3, 6)$✓

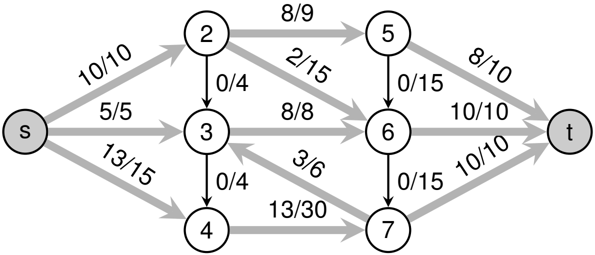
~~$(6, t)$~~

$(7, t)$✓



## Lower-Binding Edge (First Attempt)

- An edge $e \in E$ is lower-binding if decreasing its capacity by 1 also decreases the maximum flow in $G$.



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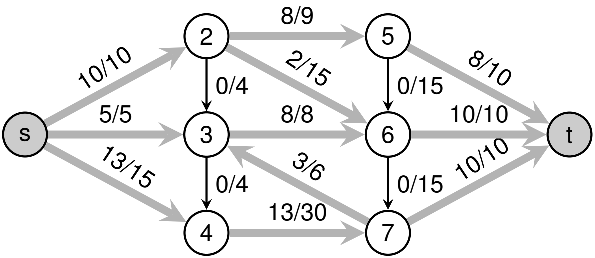

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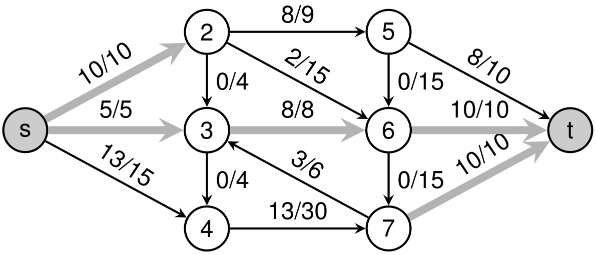

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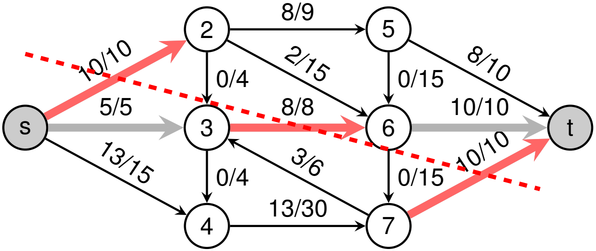

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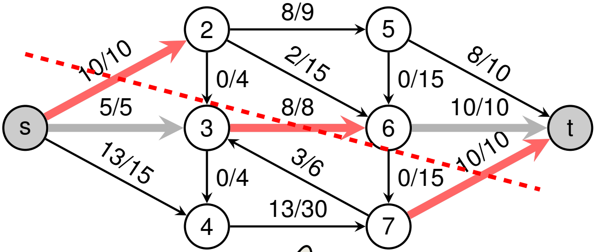


Each edge of any minimum cut is lower-binding (in fact these sets are equal due to the Max-Flow Min-Cut Theorem!)

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- An edge $e \in E$ is lower-binding if decreasing its capacity by 1 also decreases the maximum flow in $G$.

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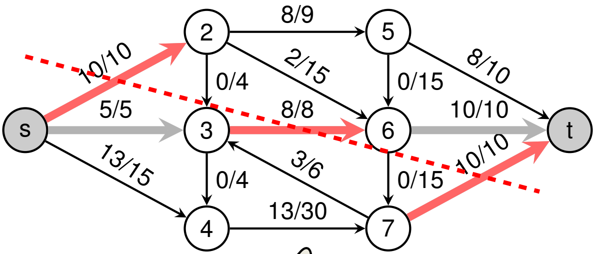
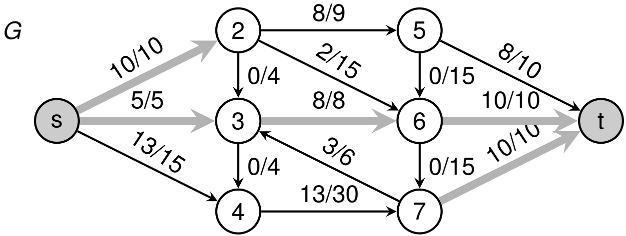



Each edge of any minimum cut is lower-binding (in fact these sets are equal due to the Max-Flow Min-Cut Theorem!)

Given the maximum flow, we know **one** minimum cut (see proof of Key Lemma) but there could be other minimum cuts!

# Lower-Binding Edge (Second Attempt)

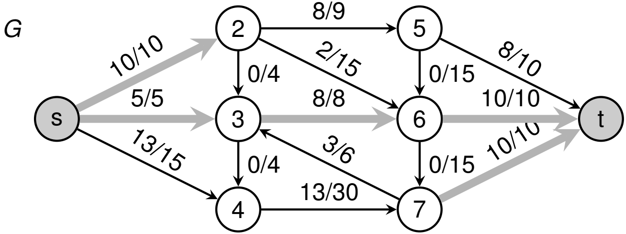
- **Idea:** An edge $(u, v)$ is **not** lower-binding iff there is a “way” to reroute the flow




$(s, 2)$✓
$(3, 6)$✓
$(7, t)$✓

## Lower-Binding Edge (Second Attempt)

- **Idea:** An edge $(u, v)$ is **not** lower-binding iff there is a "way" to reroute the flow (way $=$ path from $u$ to $v$ in the residual graph)




$(s, 2)$✓

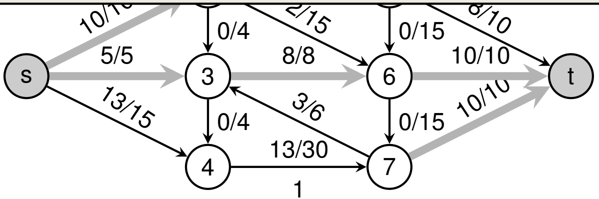
$(3, 6)$✓

$(7, t)$✓

# Lower-Binding Edge (Second Attempt)

- **Idea:** An edge $(u, v)$ is **not** lower-binding iff there is a “way” to reroute the flow (way $=$ path from $u$ to $v$ in the residual graph)

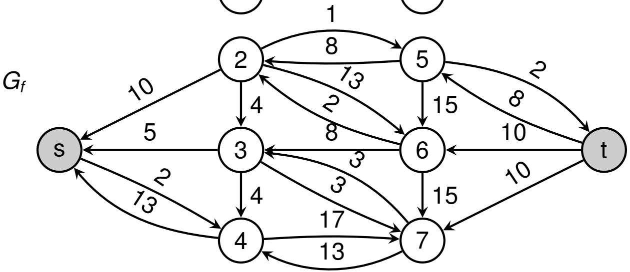
Not completely obvious, requires a proof (which is not given here)




$(s, 2)$✓

$(3, 6)$✓

$(7, t)$✓



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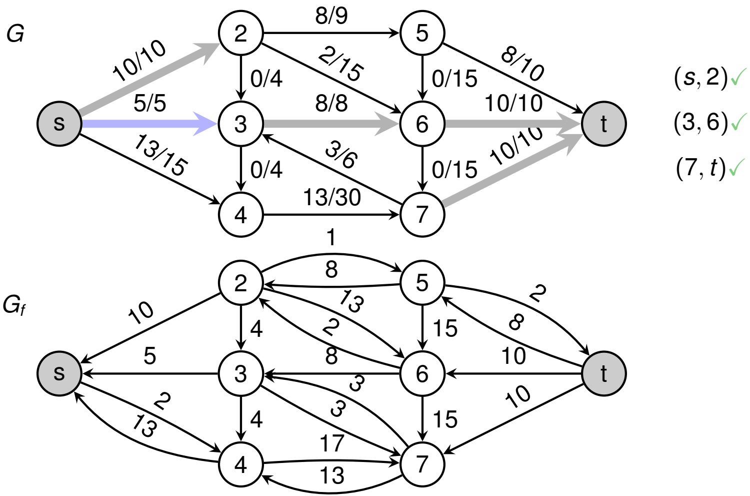

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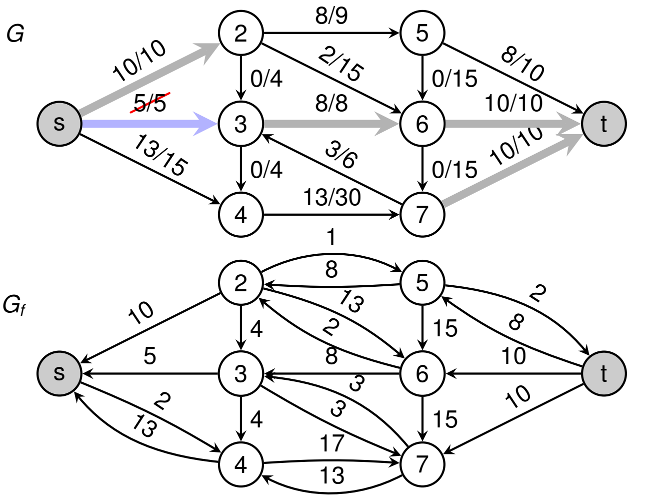


$(s, 2)$✓

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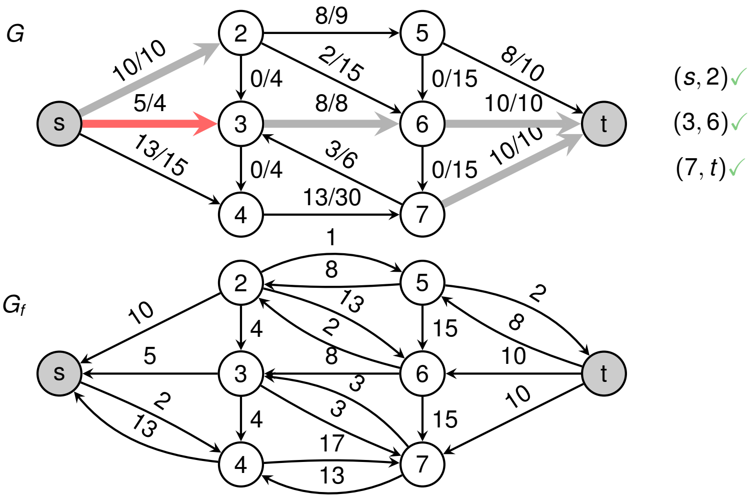

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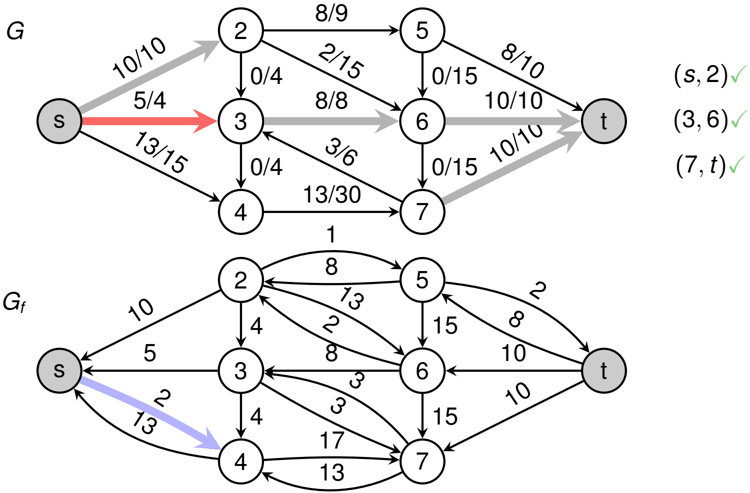

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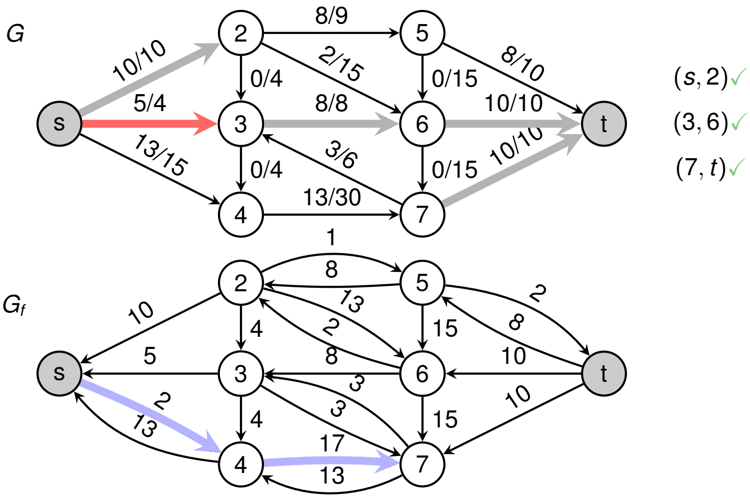

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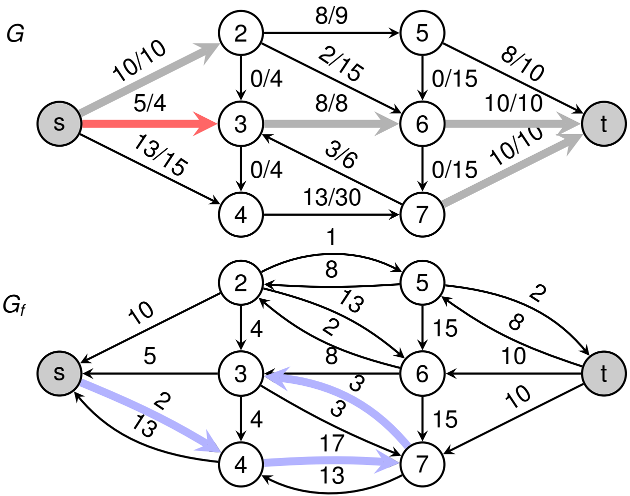


$(s, 2)$✓

$(3, 6)$✓

$(7, t)$✓

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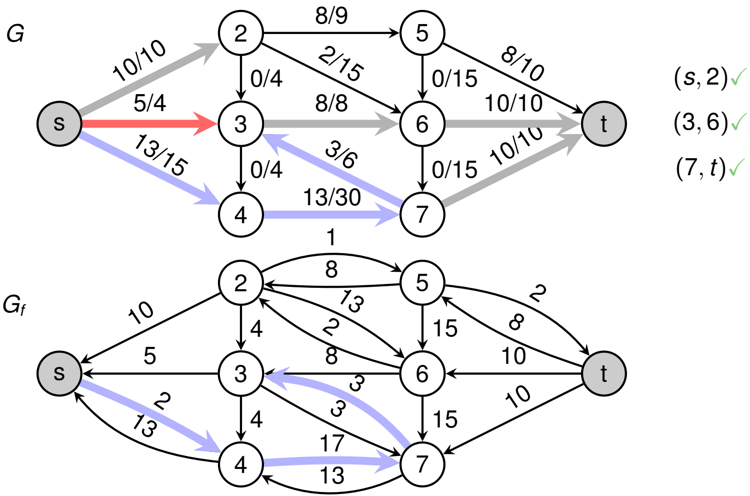

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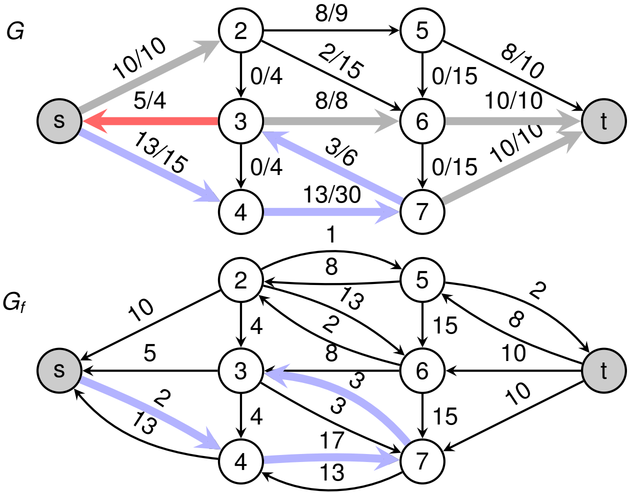


$(s, 2)$✓

$(3, 6)$✓

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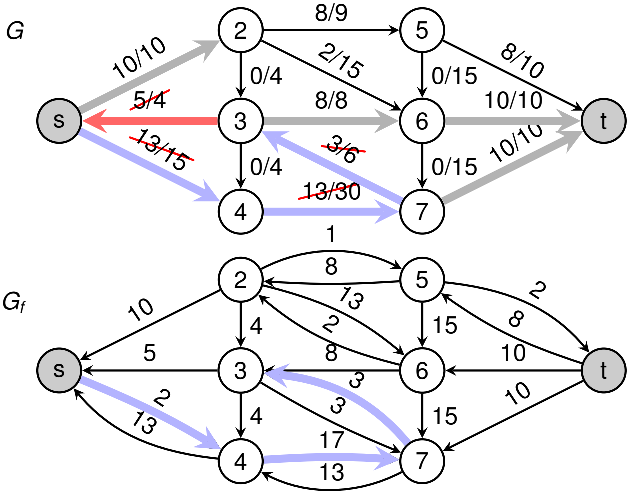


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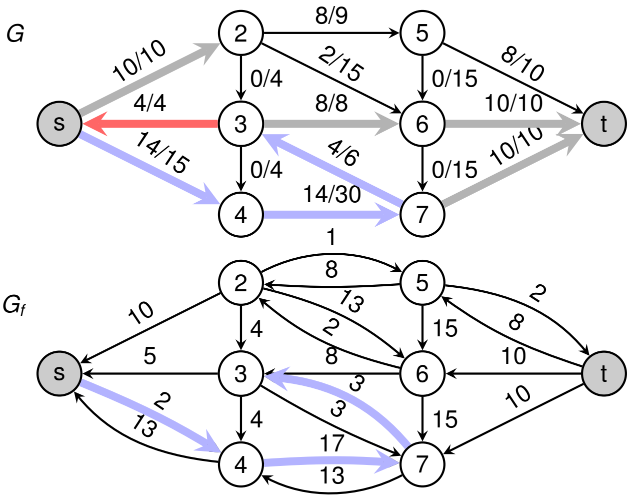


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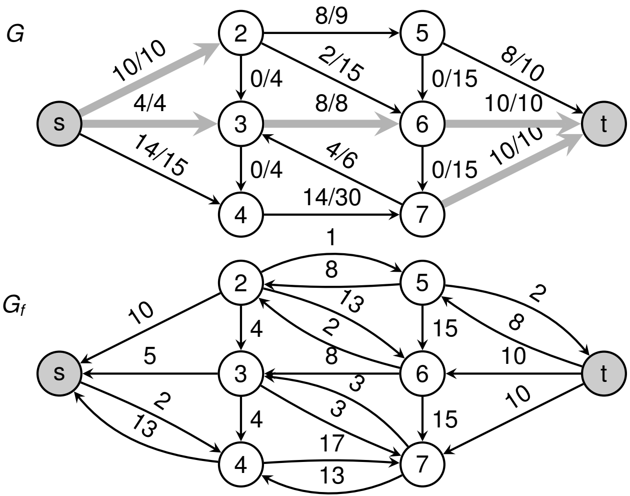


$(s,2)$✓

$(3,6)$✓

$(7,t)$✓

~~$(s,3)$~~

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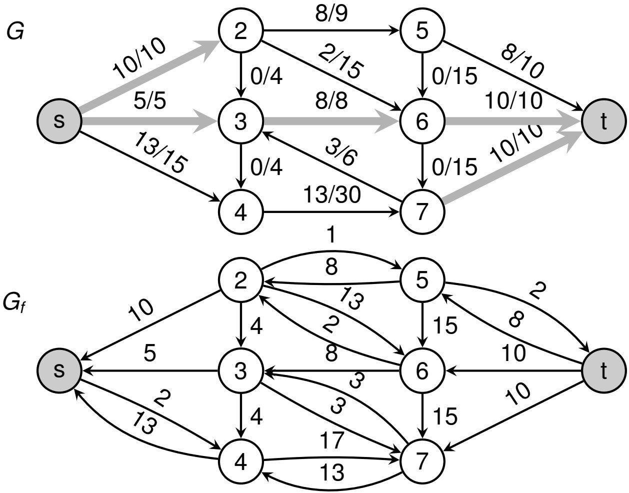


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$(7, t)$ ✓

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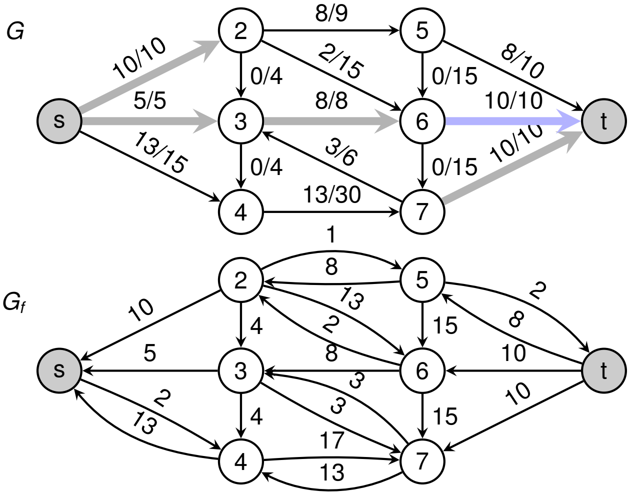


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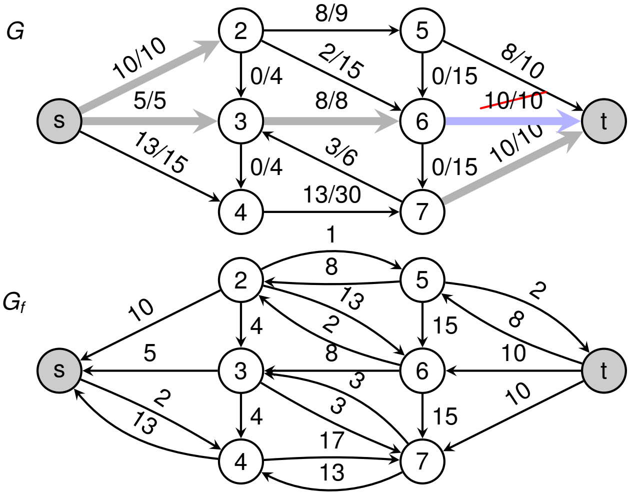


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$(3, 6)$ ✓

$(7, t)$ ✓

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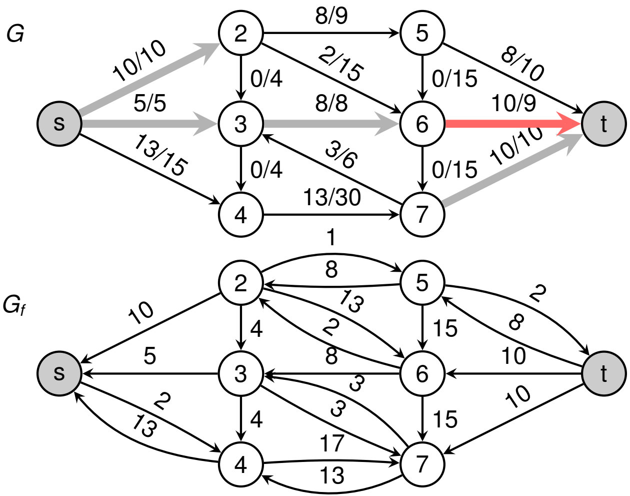


$(s, 2)$ ✓

$(3, 6)$ ✓

$(7, t)$ ✓

$\cancel{(s, 3)}$

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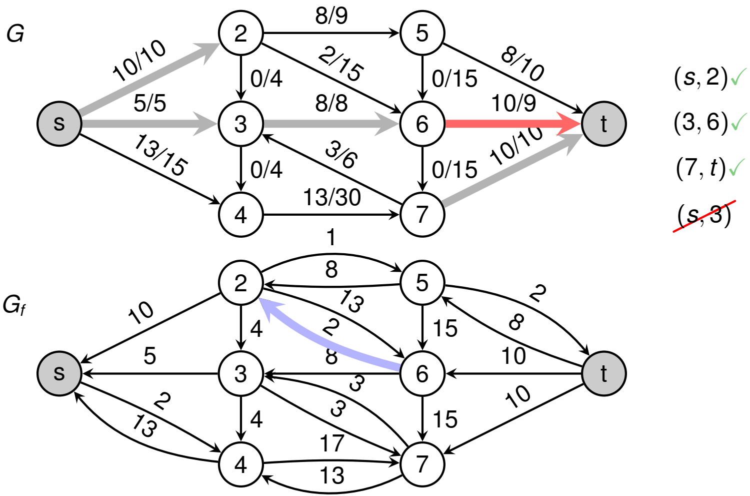

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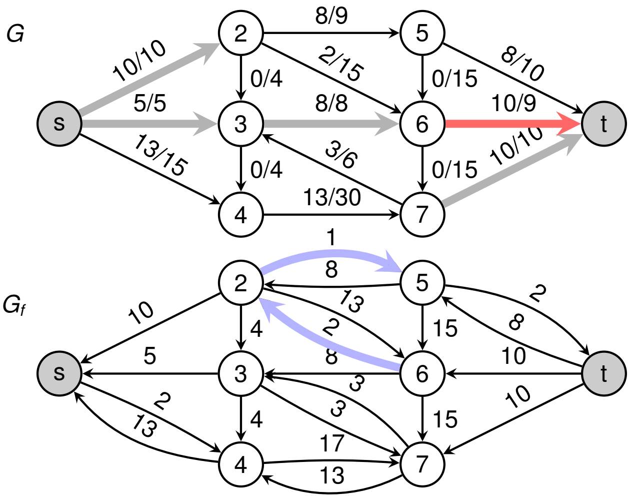


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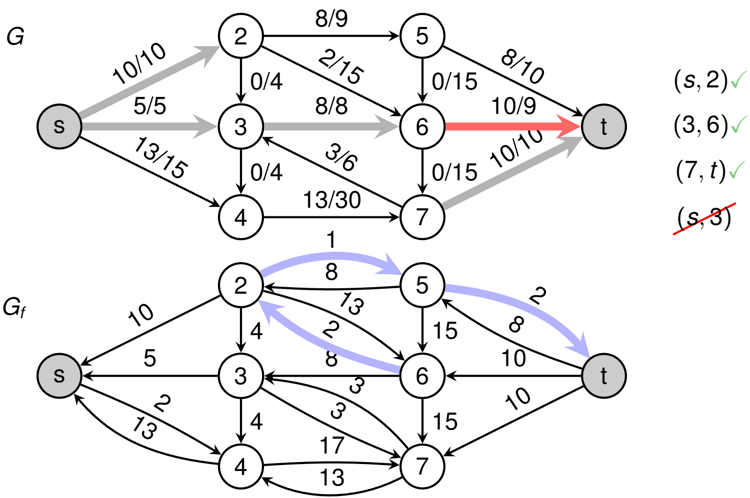

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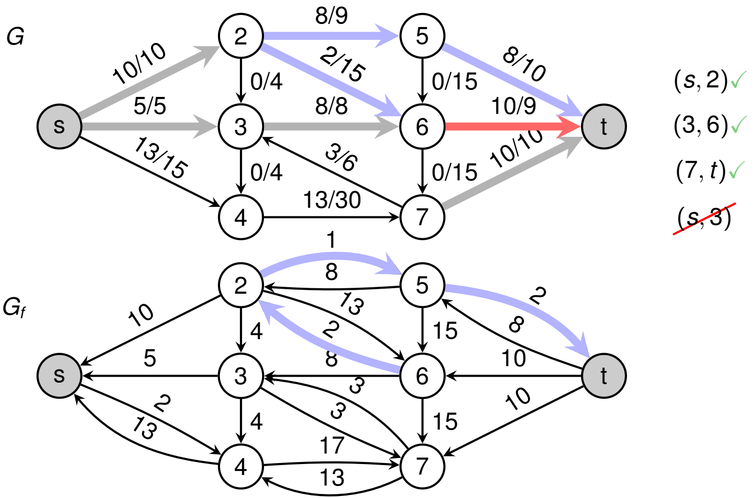

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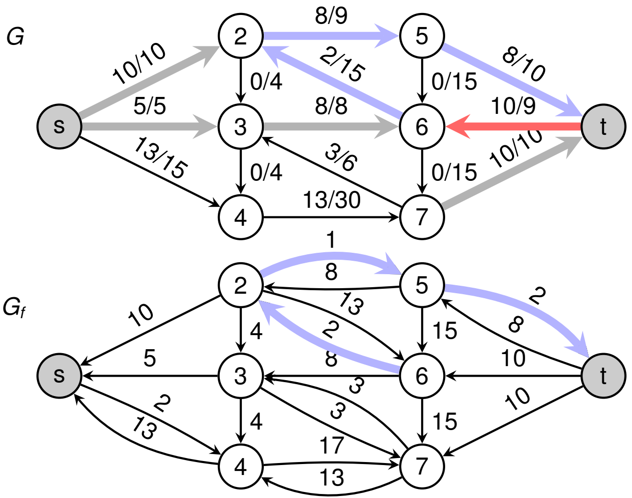


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$(7, t)$✓

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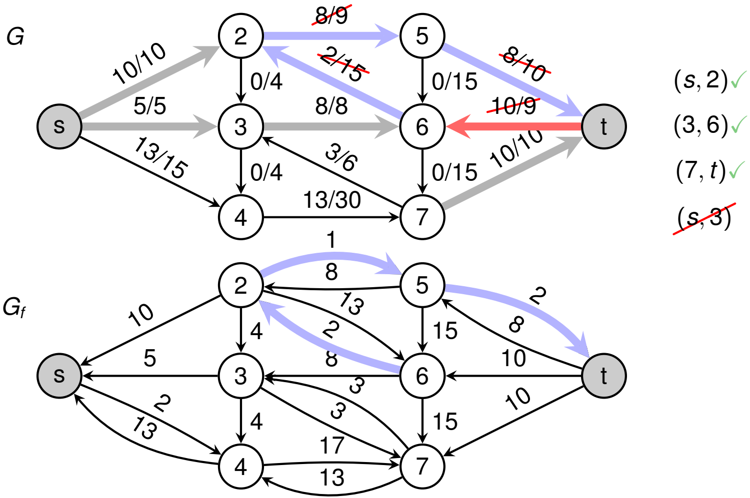

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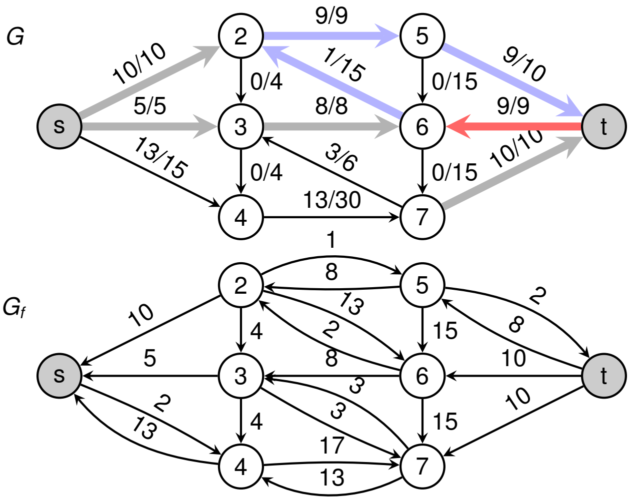


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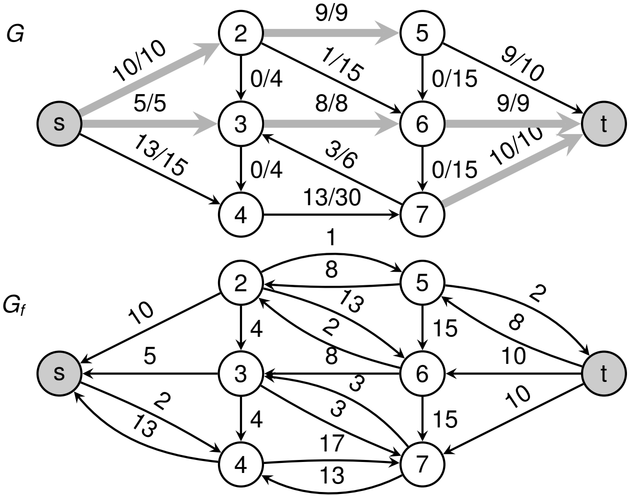


$(s, 2)$ ✓

$(3, 6)$ ✓

$(7, t)$ ✓

~~$(s, 3)$~~

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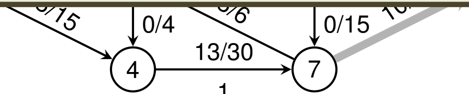


$(s, 2)$ ✓

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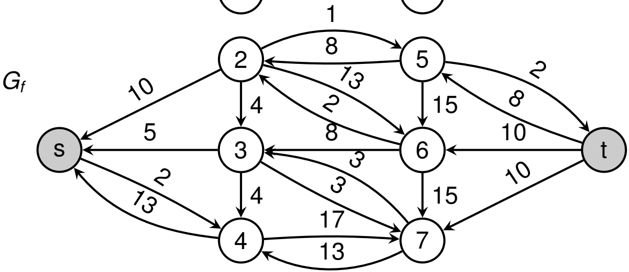
$\cancel{(6, t)}$



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- **Idea:** An edge $(u, v)$ is **not** lower-binding iff there is a "way" to reroute the flow (way $=$ path from $u$ to $v$ in the residual graph)

Algorithm:

- For each candidate $e = (u, v)$ check if $\exists$ path from $u$ to $v$ in $G_f$

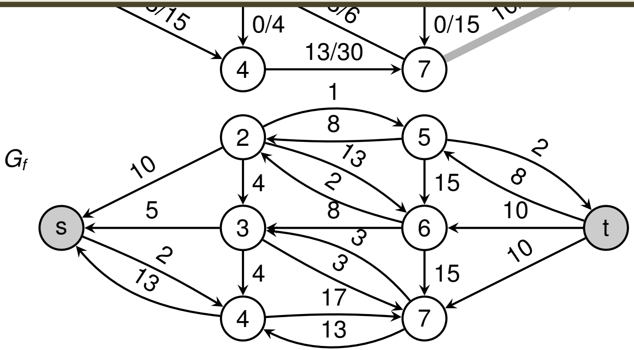
$(s, 2)$✓

$(3, 6)$✓

$(7, t)$✓

$\cancel{(s, 3)}$

$\cancel{(6, t)}$



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Algorithm:

- For each candidate $e = (u, v)$ check if $\exists$ path from $u$ to $v$ in $G_f$
- If yes, edge $(u, v)$ is not lower-binding, otherwise it is.

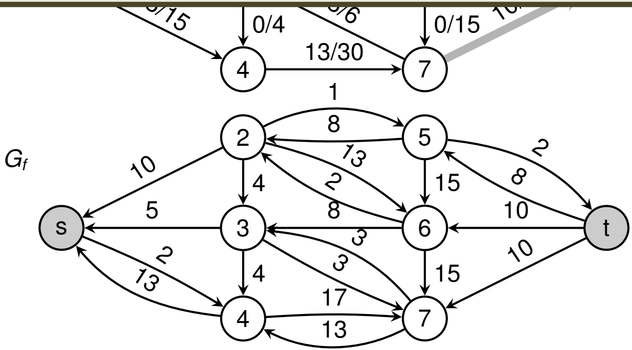
$(s, 2)$ ✓

$(3, 6)$ ✓

$(7, t)$ ✓

~~$(s, 3)$~~

~~$(6, t)$~~



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Algorithm:

- For each candidate $e = (u, v)$ check if $\exists$ path from $u$ to $v$ in $G_f$
- If yes, edge $(u, v)$ is not lower-binding, otherwise it is.

$\rightsquigarrow$ $E$ times DFS/BFS $\Rightarrow$ Runtime $O(E \cdot (V + E))$

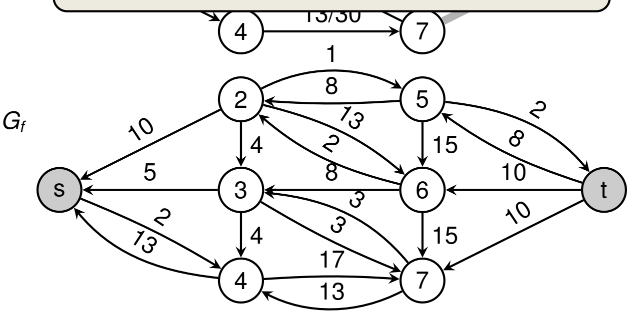
$(s, 2)$ ✓

$(3, 6)$ ✓

$(7, t)$ ✓

~~$(s, 3)$~~

~~$(6, t)$~~



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- If yes, edge $(u, v)$ is not lower-binding, otherwise it is.

$\rightsquigarrow$ $E$ times DFS/BFS $\Rightarrow$ Runtime $O(E \cdot (V + E))$

Tweak: Only need two DFS/BFS per vertex $\Rightarrow$ Runtime $O(V \cdot (V + E))$

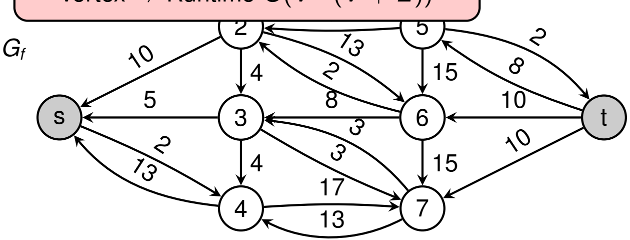
$(s, 2)$ ✓

$(3, 6)$ ✓

$(7, t)$ ✓

$\cancel{(s, 3)}$

$\cancel{(6, t)}$



# Lower-Binding Edge (Second Attempt)

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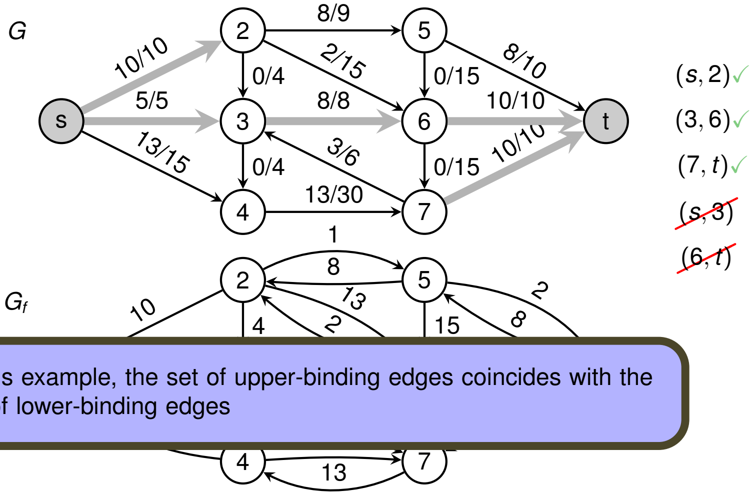


In this example, the set of upper-binding edges coincides with the set of lower-binding edges

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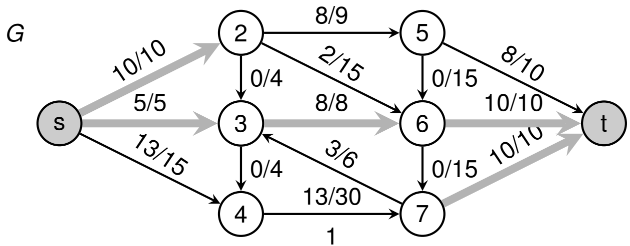


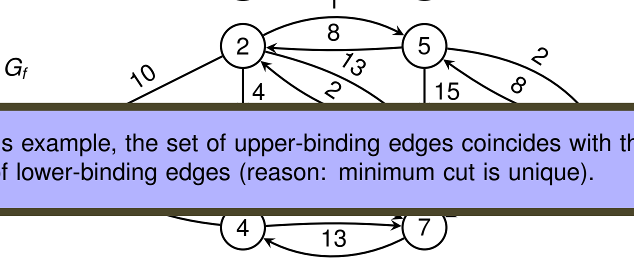

$(s, 2)$ ✓

$(3, 6)$ ✓

$(7, t)$ ✓

$\cancel{(s, 3)}$

$\cancel{(6, t)}$




In this example, the set of upper-binding edges coincides with the set of lower-binding edges (reason: minimum cut is unique).