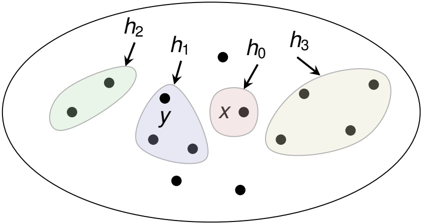


5.3: Disjoint Sets

Frank Stajano

Thomas Sauerwald

Lent 2015

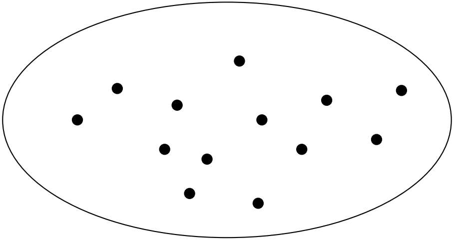
Outline

Disjoint Sets

Introduction to Graphs and Graph Searching

Disjoint Sets (aka Union Find)

Disjoint Sets Data Structure



Disjoint Sets (aka Union Find)

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- `Handle MakeSet(Item x)`
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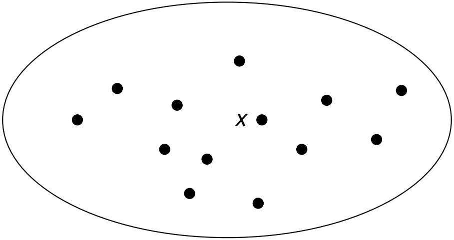
h_0=`makeSet(x)`



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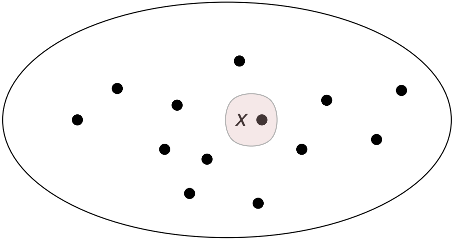

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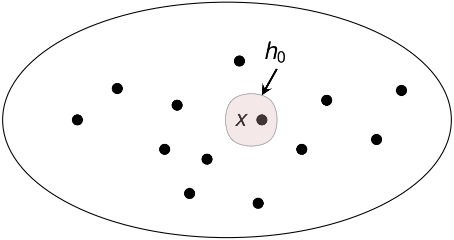

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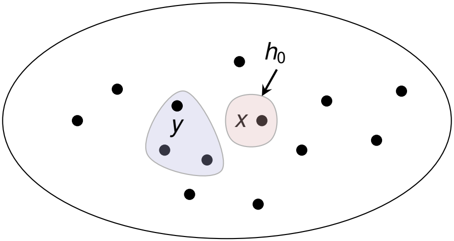

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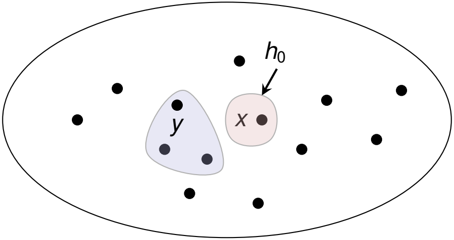

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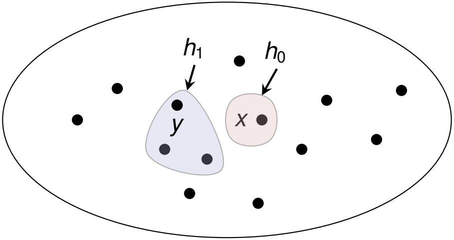

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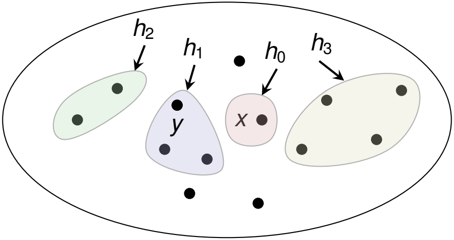

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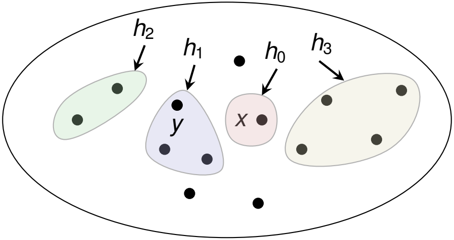

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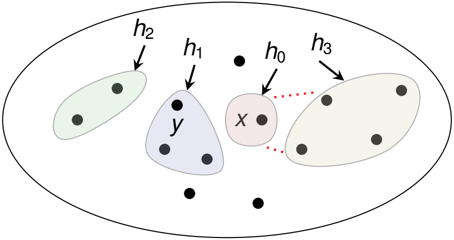

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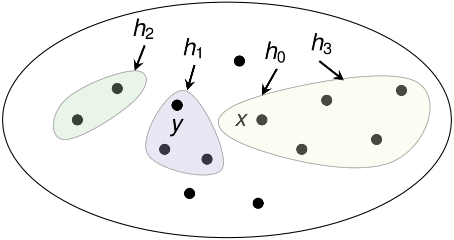

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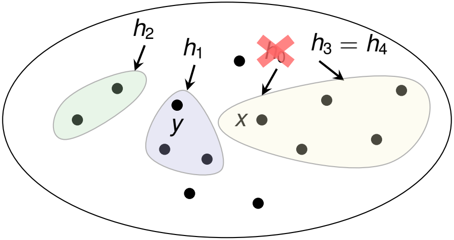

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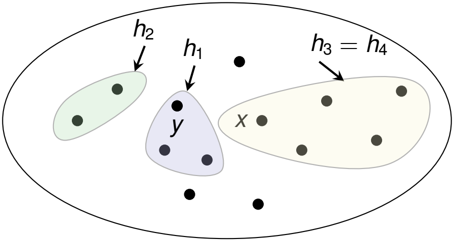

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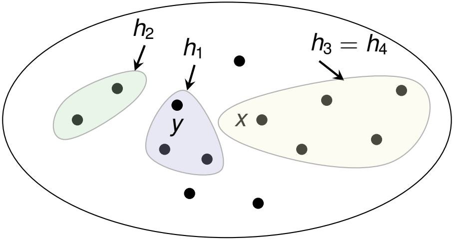

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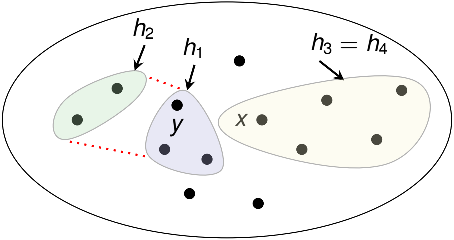

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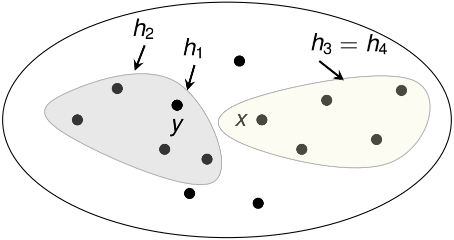

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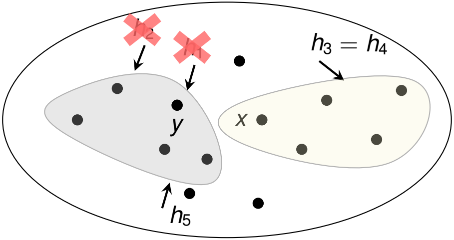

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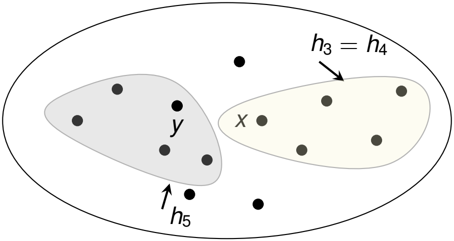
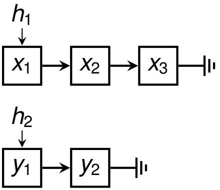


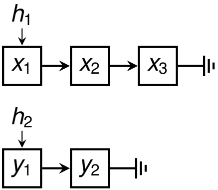
First Attempt: List Implementation



First Attempt: List Implementation

UNION-Operation

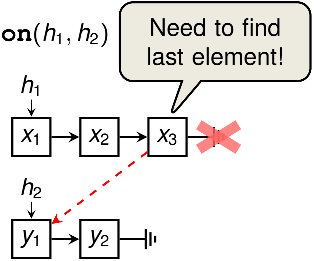
`Union`(h_1, h_2)



First Attempt: List Implementation

UNION-Operation

$\texttt{Union}(h_1, h_2)$



First Attempt: List Implementation

Union-Operation

- Add extra pointer to the last element in each list

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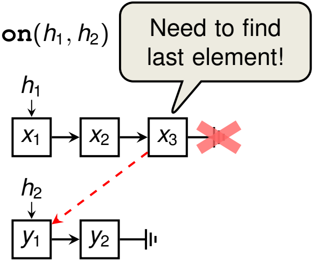

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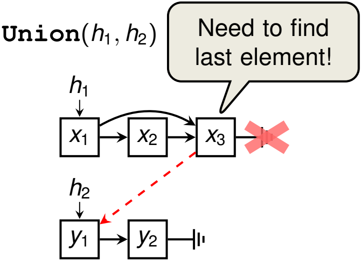

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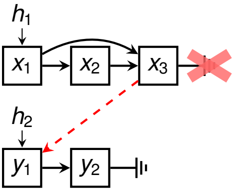
UNION-Operation

- Add extra pointer to the last element in each list

$\Rightarrow$ UNION takes constant time

Union(h_1, h_2)



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FIND-Operation

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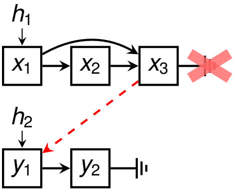

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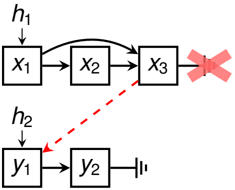
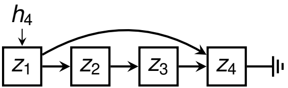


FIND-Operation

$\texttt{FindSet}(z_3)$



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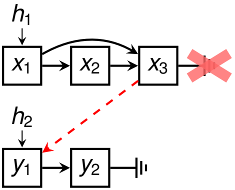
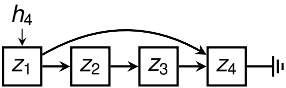


FIND-Operation

- Add backward pointer to the list head from everywhere

$\texttt{FindSet}(z_3)$



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$\Rightarrow$ Union takes constant time

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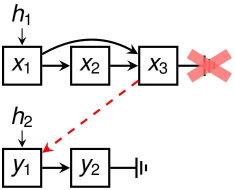

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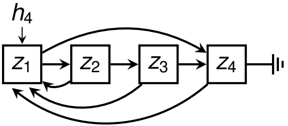

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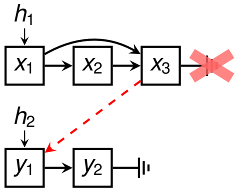
$\Rightarrow$ UNION takes constant time

`Union`(h_1, h_2)



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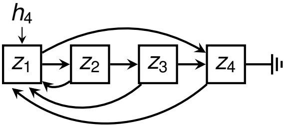

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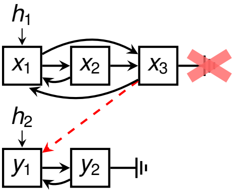

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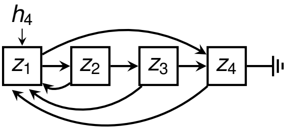
$\Rightarrow$ FIND takes constant time

FindSet(z_3)



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$\Rightarrow$ Union takes constant time

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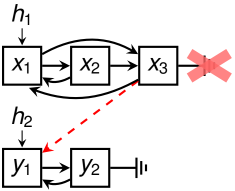
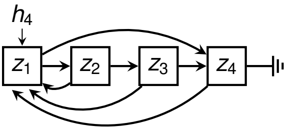


Need to update all backward pointers!

FindSet(z_3)



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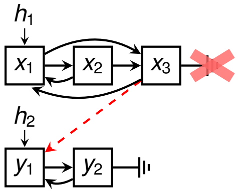
$\Rightarrow$ ~~UNION takes constant time~~

$\texttt{Union}(h_1, h_2)$



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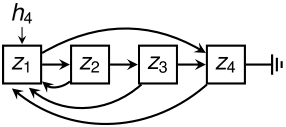

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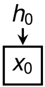

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$h_0 = d.\texttt{\textbf{Union}}(h_1, h_0)$

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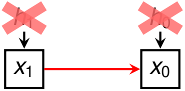

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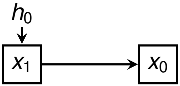

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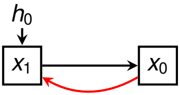

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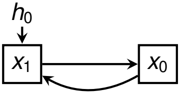

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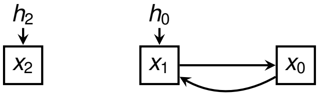

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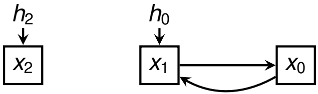

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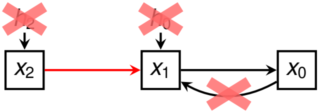

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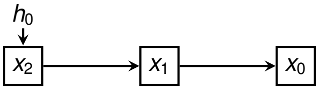

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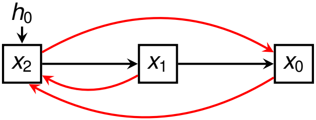

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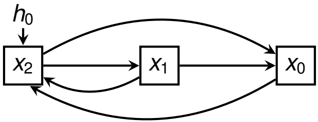

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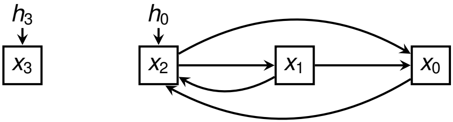

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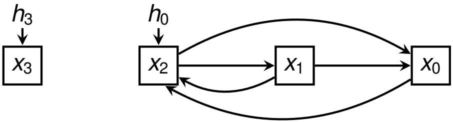

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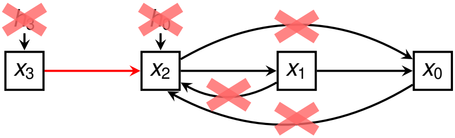

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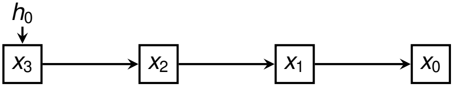

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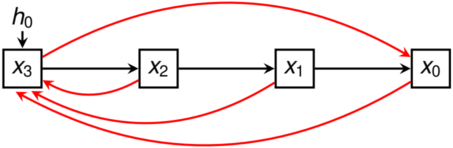

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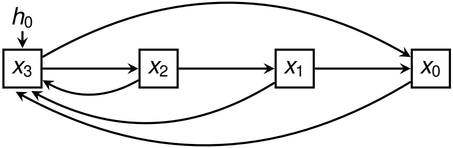

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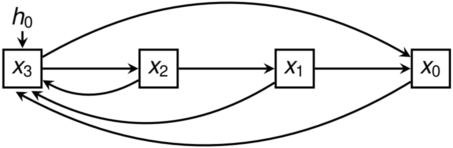

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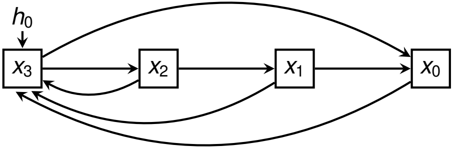

better to append shorter list to longer $\rightsquigarrow$ Weighted-Union Heuristic

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Amortized Analysis: Every operation has amortized cost $\mathcal{O}(\log n)$, but there may be operations with total cost $\Theta(n)$.

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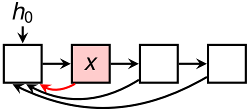

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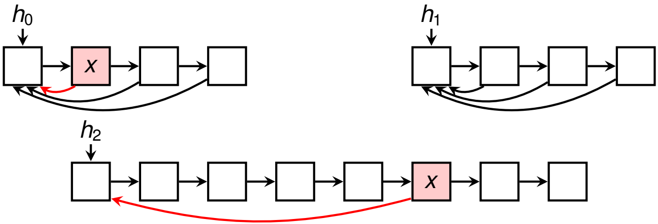

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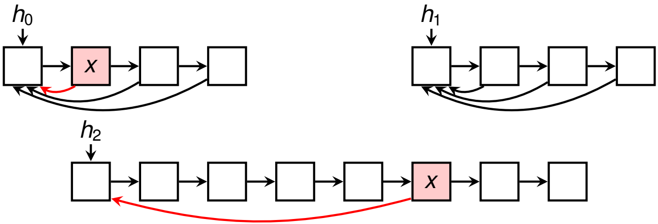

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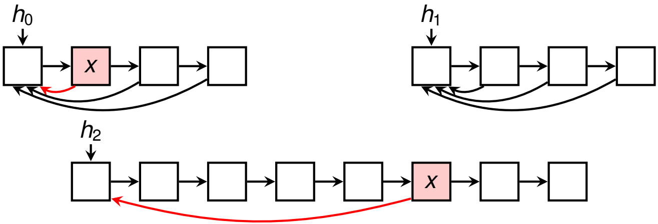

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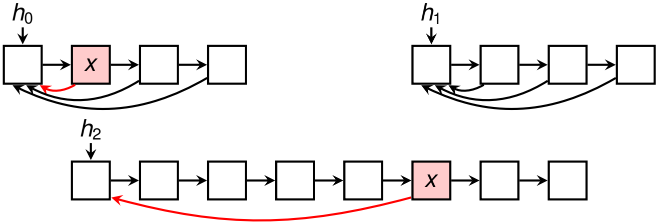

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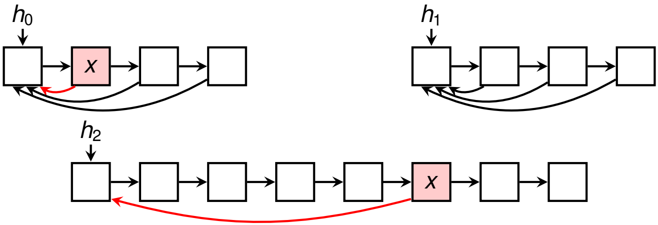

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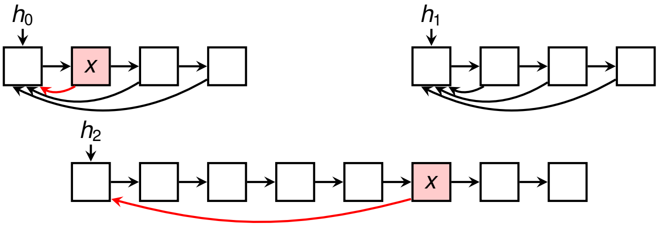

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Can we improve on this further?

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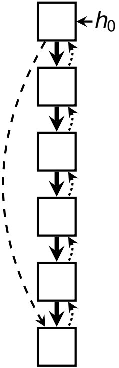
How to Improve?



Doubly-Linked List

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- UNION: $\mathcal{O}(1)$

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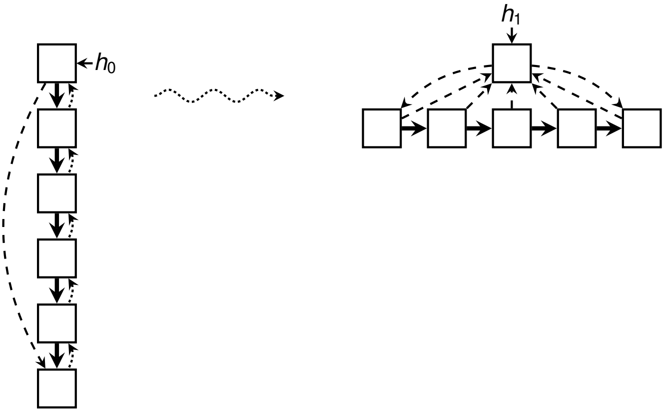

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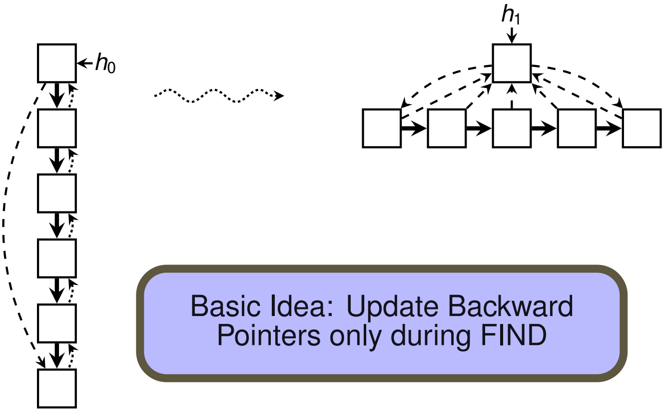

Basic Idea: Update Backward Pointers only during FIND

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Weighted-Union Heuristic

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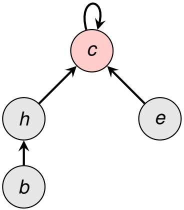
Disjoint Sets via Forests

Forest Structure

- Set is represented by a rooted tree with root being the representative
- Every node has pointer $.p$ to its parent (for root x, $x.p = x$)

Disjoint Sets via Forests

$\{b, c, e, h\}$



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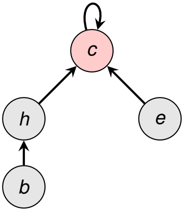

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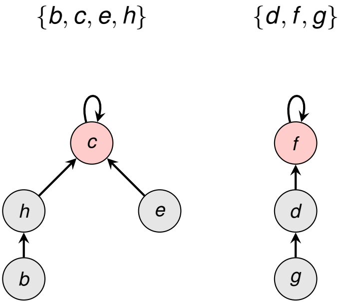

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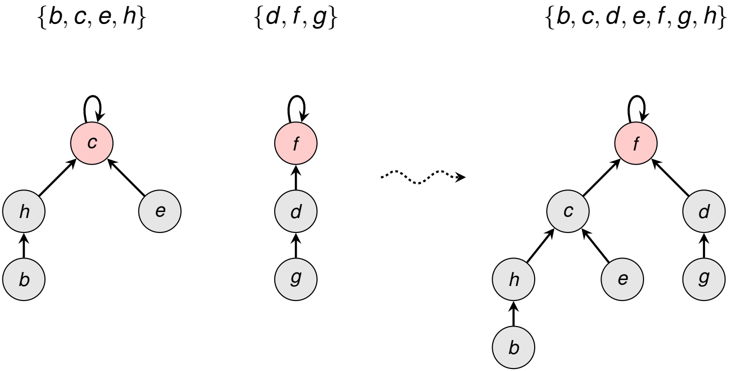

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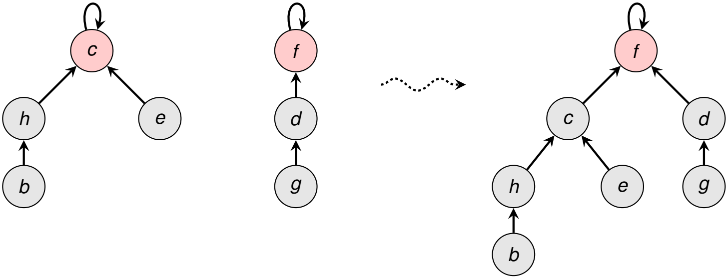

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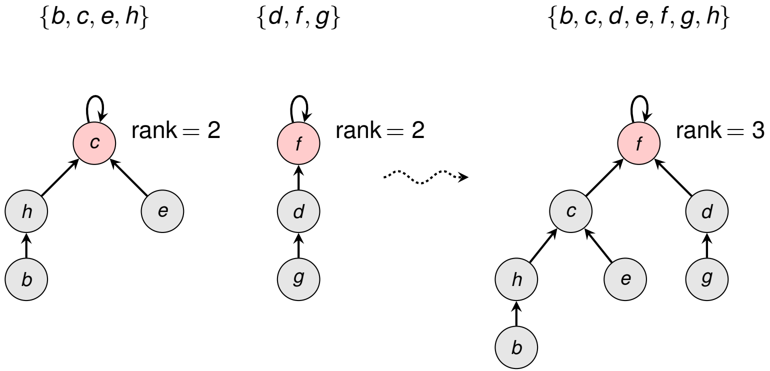

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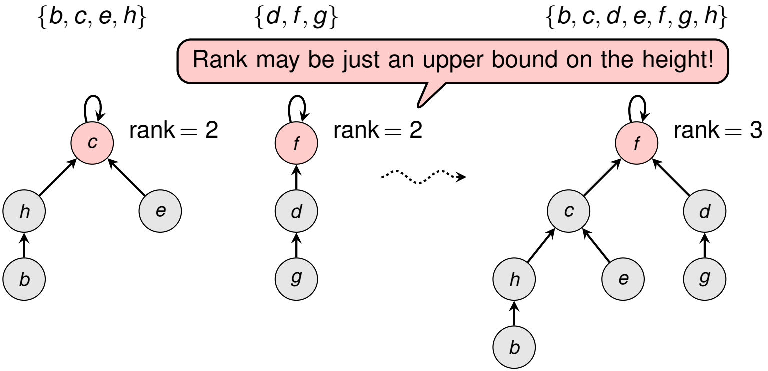

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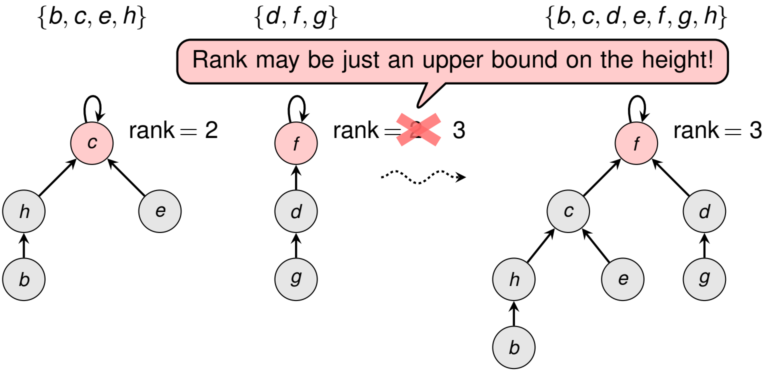
- UNION: Merge the two trees

Append tree of smaller height $\rightsquigarrow$ Union by Rank

Disjoint Sets via Forests



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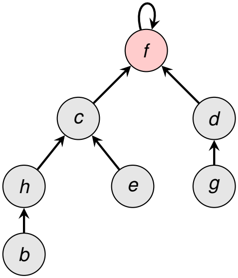
Path Compression during FIND-SET

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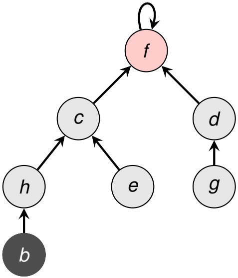
$\boxed{b}$

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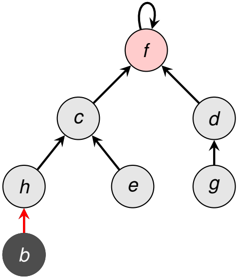


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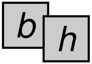

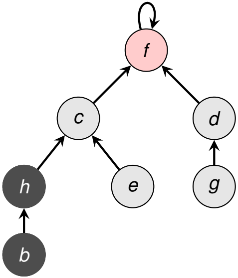


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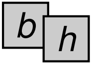

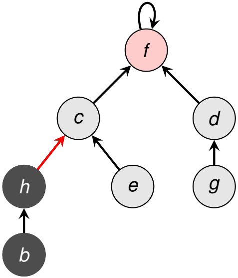


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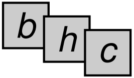

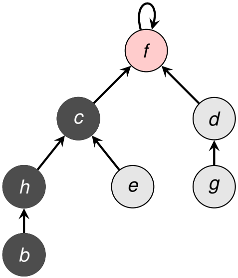


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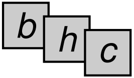

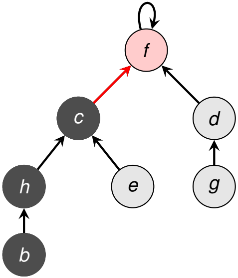


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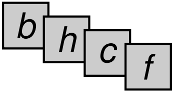

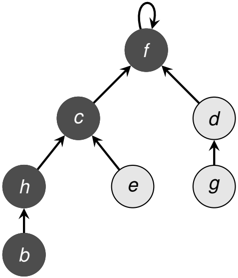


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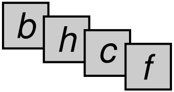

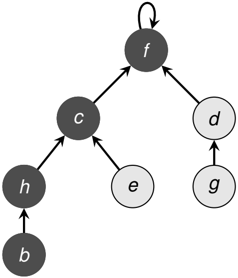


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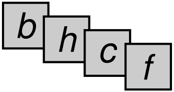

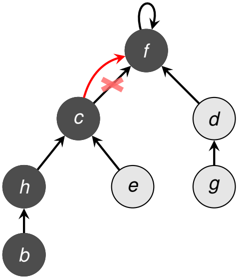


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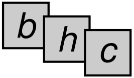

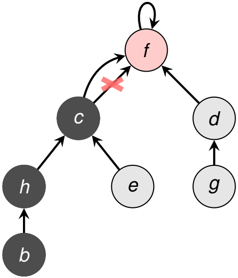


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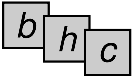

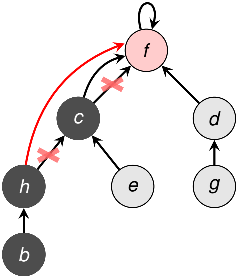


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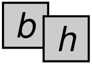

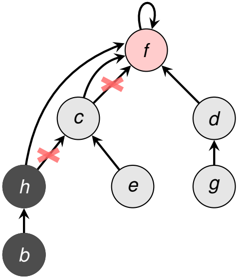


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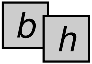

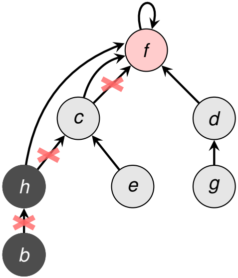


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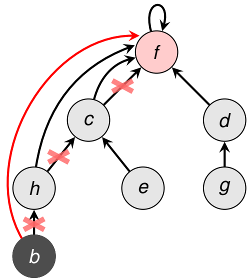


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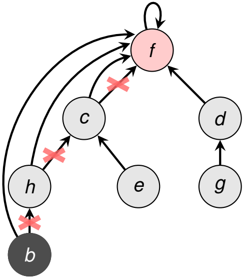


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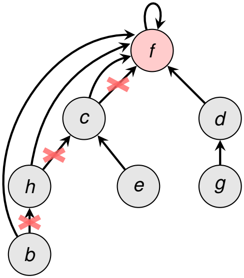


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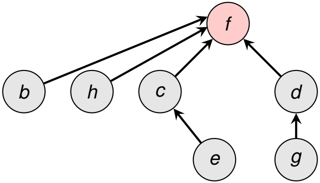


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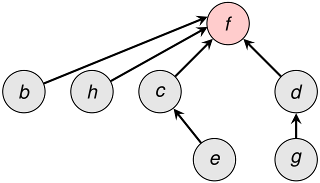


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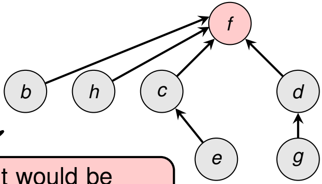

Maintaining the exact height would be costly, hence rank is only an **upper bound**!

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Combining Union by Rank and Path Compression

Theorem 21.14

Any sequence of m MAKE-SET, UNION, FIND-SET operations, n of which are MAKE-SET operations, can be performed in $\mathcal{O}(m \cdot \alpha(n))$ time.

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$$\alpha(n) = \begin{cases} 0 & \text{for } 0 \leq n \leq 2, \\ 1 & \text{for } n = 3, \\ 2 & \text{for } 4 \leq n \leq 7, \\ 3 & \text{for } 8 \leq n \leq 2047, \\ 4 & \text{for } 2048 \leq n \leq 10^{80} \end{cases}$$

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More than the number of atoms in the universe!

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$\log^*(n)$, **the iterated logarithm**, satifies $\alpha(n) \leq \log^*(n)$, but still $\log^*(10^{80}) = 5$.

Combining Union by Rank and Path Compression

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Combining Union by Rank and Path Compression

Data Structure is essentially optimal! (for more details see CLRS)

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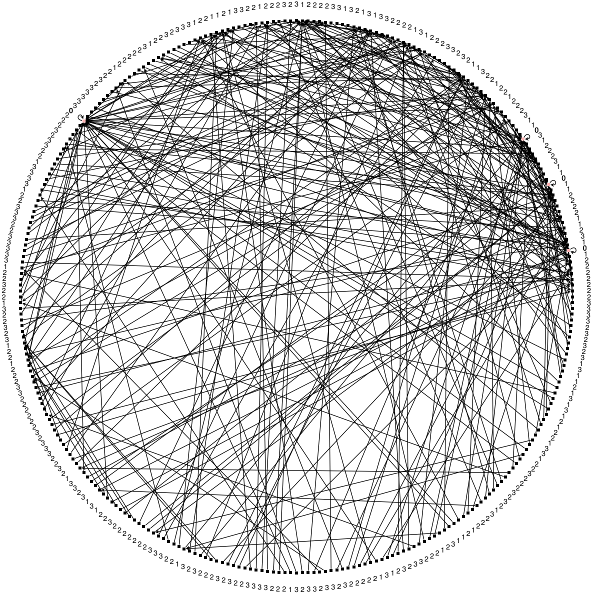
Simulating the Effects of Union by Rank and Path Compression

Simulating the Effects of Union by Rank and Path Compression

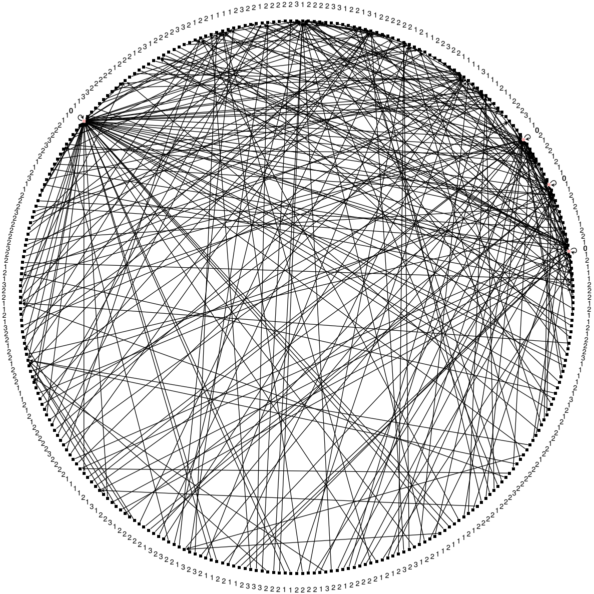
Experimental Setup

1. Initialise singletons $1, 2, \ldots, 300$
2. For every $1 \leq i \leq 300$, pick a random $1 \leq r \leq 300$, $r \neq i$ and perform $\textsc{Union}(\textsc{Find}(i), \textsc{Find}(r))$
3. Perform $j \in \{0, 100, 200, 300, 600, 900\}$ many $\textsc{Find}(r)$, where $1 \leq r \leq 300$ is random

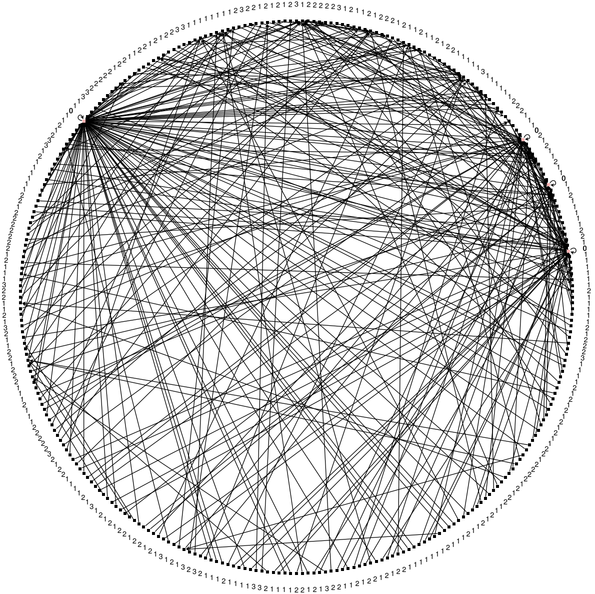
Union by Rank without Path Compression



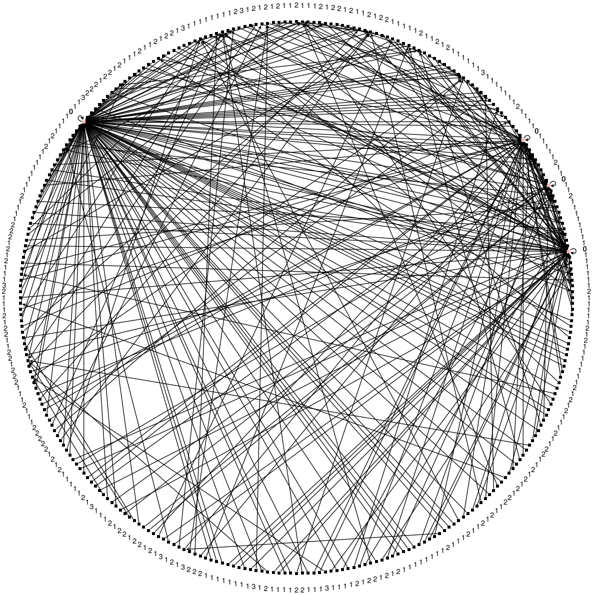
Union by Rank with Path Compression



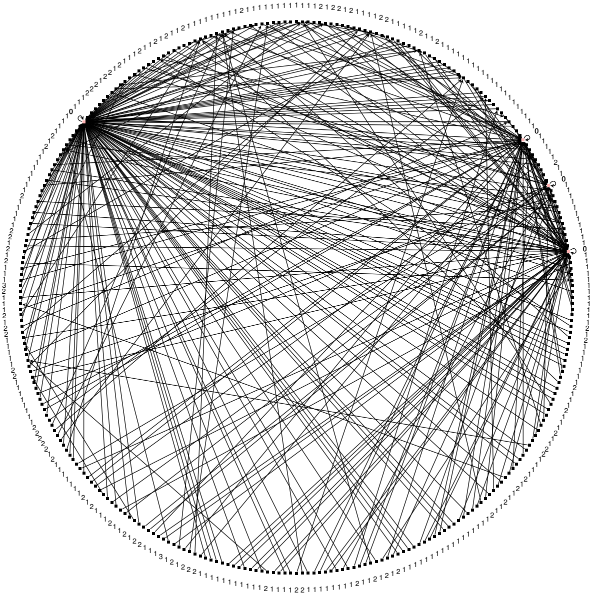
After 100 additional FINDs



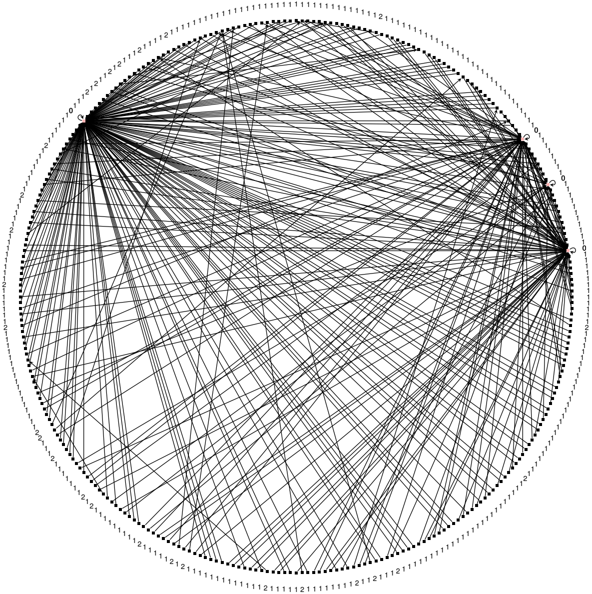
After 200 additional FINDs



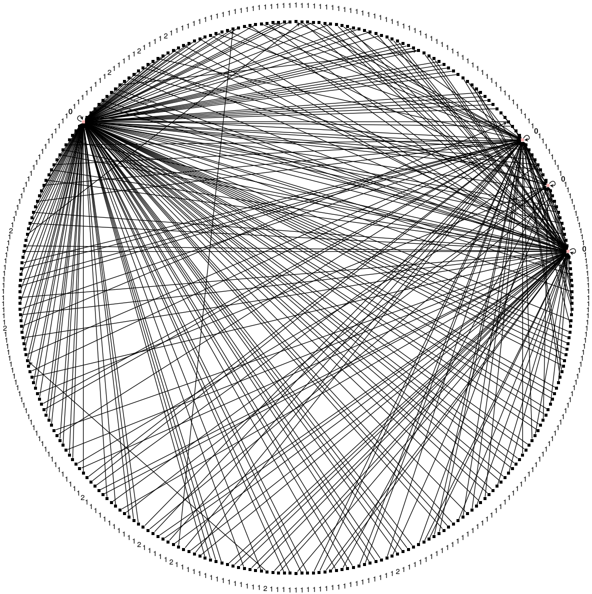
After 300 additional FINDs



After 600 additional FINDs



After 900 additional FINDs

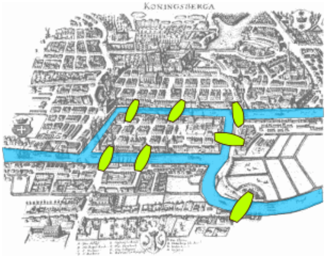


Outline

Disjoint Sets

Introduction to Graphs and Graph Searching

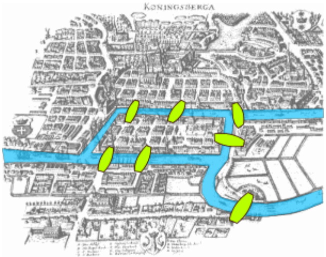
Origin of Graph Theory



Source: Wikipedia

Seven Bridges at Königsberg 1737

Origin of Graph Theory



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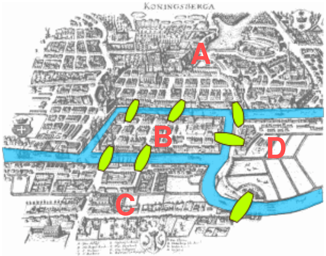
Leonhard Euler (1707-1783)

Is there a tour which crosses each bridge **exactly once**?

Origin of Graph Theory



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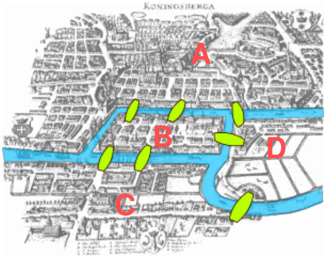

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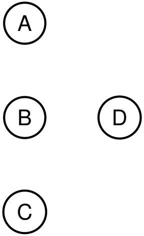

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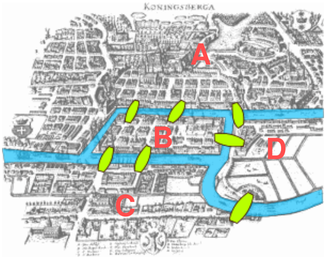

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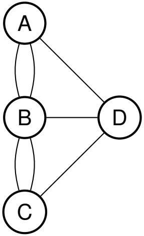

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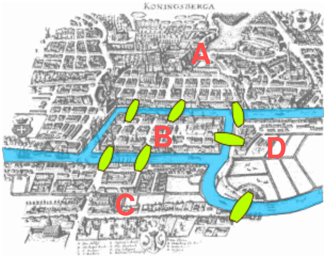

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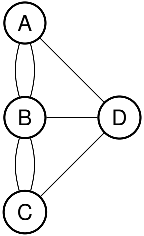

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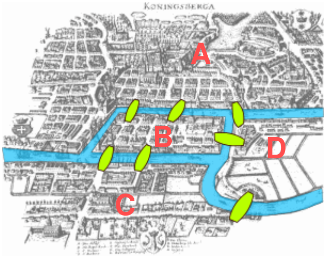

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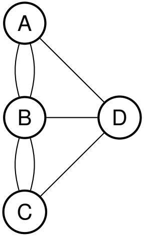

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$\rightsquigarrow$ 1B course: Complexity Theory

What is a Graph?

Directed Graph

A graph $G = (V, E)$ consists of:

- V: the set of vertices
- E: the set of edges (arcs)

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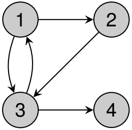

What is a Graph?

Directed Graph

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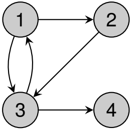

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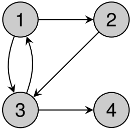

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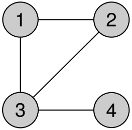

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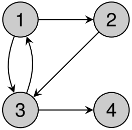

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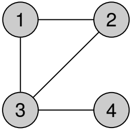

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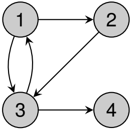

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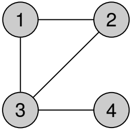

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- A sequence of edges between two vertices forms a path

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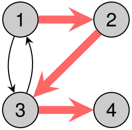
Path $p = (1, 2, 3, 4)$



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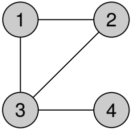

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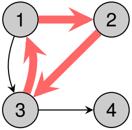
Path $p = (1, 2, 3, 1)$, which is a cycle



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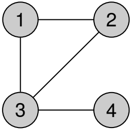

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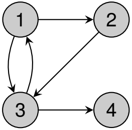

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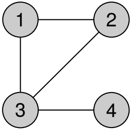

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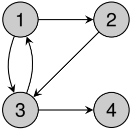

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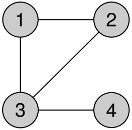

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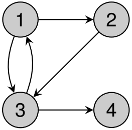

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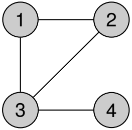

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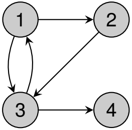

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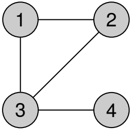

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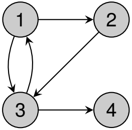

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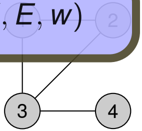
Later: edge-weighted graphs $G = (V, E, w)$



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Representations of Directed and Undirected Graphs

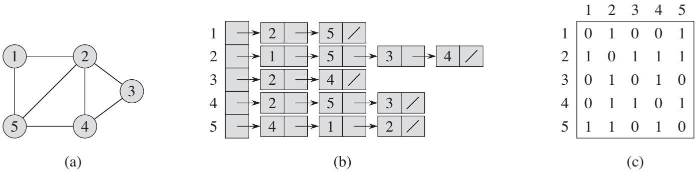

Figure 22.1 Two representations of an undirected graph. **(a)** An undirected graph G with 5 vertices and 7 edges. **(b)** An adjacency-list representation of G. **(c)** The adjacency-matrix representation of G.

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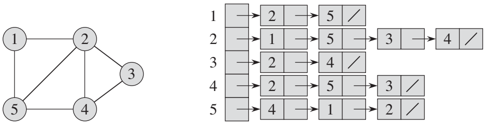

(a) (b)

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

(c)

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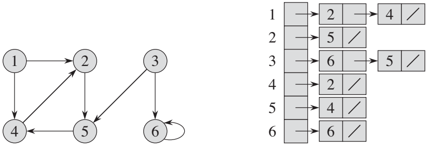

(a) (b)

	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

(c)

Figure 22.2 Two representations of a directed graph. **(a)** A directed graph G with 6 vertices and 8 edges. **(b)** An adjacency-list representation of G. **(c)** The adjacency-matrix representation of G.

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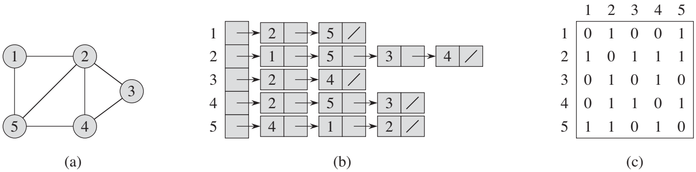

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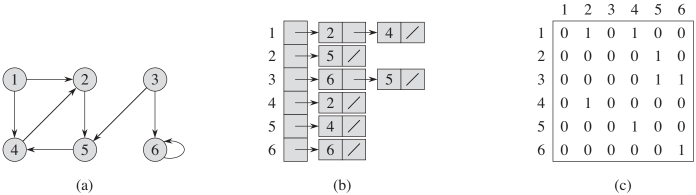

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