

Lecture 4: name abstraction

Alpha-equivalence

Recall from Lecture 1 the equivalence relation $=_\alpha$ on $Tr \triangleq \{t ::= \forall a \mid A(t, t) \mid L(a, t)\}$

$$\frac{a \in \mathbb{A}}{\forall a =_\alpha \forall a} \quad \frac{t_1 =_\alpha t'_1 \quad t_2 =_\alpha t'_2}{A(t_1, t_2) =_\alpha A(t'_1, t'_2)}$$

$$\frac{(a \ b) \cdot t =_\alpha (a' \ b) \cdot t' \quad b \notin \{a, a'\} \cup \text{var}(t \ t')}{L(a, t) =_\alpha L(a', t')}$$

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this is an instance of the
nominal sets notion of
'freshness'

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For each nominal set X , we can define a relation $\# \subseteq A \times X$ of **freshness**:

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- ▶ In \mathbb{N} , $a \# n$ always.
- ▶ In \mathbb{A} , $a \# b$ iff $a \neq b$.
- ▶ In Tr , $a \# t$ iff $a \notin \text{var } t$
- ▶ In Λ , $a \# [t]_\alpha$ iff $a \notin \text{fv } t$.
- ▶ In $X \times Y$, $a \# (x, y)$ iff $a \# x$ and $a \# y$.
- ▶ In $X \rightarrow_{\text{fs}} Y$, $a \# f$ can be subtle!
(and hence ditto for $\mathbf{P}_{\text{fs}} X$)

Freshness

For each nominal set X , we can define a relation $\# \subseteq \mathbb{A} \times X$ of freshness:

$$a \# x \triangleq a \notin \text{supp } x$$

Note that if $f \in \text{Nom}(X, Y)$, then for all $x \in X$ we have $\text{supp}(f x) \subseteq \text{supp } x$ (Lemma 1 from L3) and hence

$$a \# x \Rightarrow a \# f x$$

(More generally, if $f \in X \rightarrow_{\text{fs}} Y$ and $x \in X$, then $a \# f$ and $a \# x$ implies $a \# f x$.)

Freshness

For each nominal set X , we can define a relation $\# \subseteq \mathbb{A} \times X$ of freshness:

$$a \# x \triangleq a \notin \text{supp } x$$

Fact: $\#$ is an equivariant relation

$$(\forall \pi \in \text{Perm } \mathbb{A}) a \# x \Rightarrow \pi a \# \pi \cdot x$$

Indeed $\pi \cdot (\text{supp } x) = \text{supp}(\pi \cdot x)$

(Exercise)

[Cf. *Equivariance Principle* — NSB p21]

Name abstraction

Each $X \in \mathbf{Nom}$ yields a nominal set $[A]X$ of

name-abstractions $\langle a \rangle x$ are \sim -equivalence classes of pairs $(a, x) \in A \times X$, where

$$(a, x) \sim (a', x') \Leftrightarrow \exists b \# (a, x, a', x') \\ (b a) \cdot x = (b a') \cdot x'$$

The **Perm** A -action on $[A]X$ is well-defined by

$$\pi \cdot \langle a \rangle x = \langle \pi(a) \rangle (\pi \cdot x)$$

Lemma 2. $\text{supp}(\langle a \rangle x) = \text{supp } x - \{a\}$, so that
 $b \# \langle a \rangle x \Leftrightarrow b = a \vee b \# x$

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Proof. $\left. \begin{array}{l} \mathbb{A} \times X \rightarrow [\mathbb{A}]X \\ (a, x) \mapsto \langle a \rangle x \end{array} \right\}$ is equivariant. So by Lemma 1
from L3, $\text{supp} \langle a \rangle x \subseteq \text{supp}(a, x) = \{a\} \cup \text{supp } x$.

Lemma 2. $\text{supp}(\langle a \rangle x) = \text{supp } x - \{a\}$, so that
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Proof.

$$\text{supp} \langle a \rangle x \subseteq \{a\} \cup \text{supp } x$$

Note that $(a, x) \sim (a, x') \Rightarrow x = x'$.

So if $\pi a = a$ and $\pi \cdot \langle a \rangle x = \langle a \rangle x$, then $\langle a \rangle (\pi \cdot x) = \langle a \rangle x$ and hence $\pi \cdot x = x$.

Therefore, if A supports $\langle a \rangle x$, then $A \cup \{a\}$ supports x , and hence $\text{supp } x \subseteq \text{supp} \langle a \rangle x \cup \{a\}$.

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Proof.

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So it suffices to show that $a \# \langle a \rangle x$.

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So it suffices to show that $a \# \langle a \rangle x$.

Given a, x , pick any a' with $a' \# (a, x)$ and hence also $a' \# \langle a \rangle x$.
So $a = (a' a) \cdot a' \# (a' a) \cdot \langle a \rangle x$ (by equivariance of $\#$).

But since $a' \# (a, x)$, we get $(a', (a' a) \cdot x) \sim (a, x)$ (check).
So $(a' a) \cdot \langle a \rangle x = \langle a' \rangle ((a' a) \cdot x) = \langle a \rangle x$.

Therefore $a \# \langle a \rangle x$. \square

Name abstraction

Each $X \in \mathbf{Nom}$ yields a nominal set $[\mathbb{A}]X$ of

name-abstractions $\langle a \rangle x$ are \sim -equivalence classes of pairs $(a, x) \in \mathbb{A} \times X$, where

$$(a, x) \sim (a', x') \Leftrightarrow \exists b \# (a, x, a', x') \\ (b a) \cdot x = (b a') \cdot x'$$

We get a functor $[\mathbb{A}](-) : \mathbf{Nom} \rightarrow \mathbf{Nom}$ sending $f \in \mathbf{Nom}(X, Y)$ to $[\mathbb{A}]f \in \mathbf{Nom}([\mathbb{A}]X, [\mathbb{A}]Y)$ where

$$[\mathbb{A}]f (\langle a \rangle x) = \langle a \rangle (f x)$$

Name abstraction

$[A](-) : \mathbf{Nom} \rightarrow \mathbf{Nom}$ is a kind of (affine) function space—it is right adjoint to the functor $(-)*A : \mathbf{Nom} \rightarrow \mathbf{Nom}$ sending X to $X*A = \{(x, a) \mid a \# x\}$. (Exercise)

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$[\mathbb{A}](-) : \mathbf{Nom} \rightarrow \mathbf{Nom}$ is a kind of (affine) function space—it is right adjoint to the functor $(-) * \mathbb{A} : \mathbf{Nom} \rightarrow \mathbf{Nom}$ sending X to $X * \mathbb{A} = \{(x, a) \mid a \# x\}$. (Exercise)

That explains what morphisms *into* $[\mathbb{A}]X$ look like. More important is the following characterization of morphisms *out of* $[\mathbb{A}]X$ in terms of a ‘freshness condition for binders’ [NSB p69]...

Theorem 2. If $f \in \mathbf{Nom}(X \times \mathbb{A} \times Y, Z)$ satisfies

$$(\forall x \in X, a \in \mathbb{A}, y \in Y) a \# x \Rightarrow a \# f(x, a, y) \quad (\text{FCB})$$

then there is a unique $\bar{f} \in \mathbf{Nom}(X \times [\mathbb{A}]Y, Z)$ satisfying

$$(\forall x \in X, a \in \mathbb{A}, y \in Y) a \# x \Rightarrow \bar{f}(x, \langle a \rangle y) = f(x, a, y)$$

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For example, $X = \mathbf{1} = \{0\}$, $Y = \mathbb{A}$, $Z = \mathbb{A} + \mathbf{1} = \mathbb{A} \cup \{0\}$

and $f(0, a, a') = \begin{cases} 0 & \text{if } a = a' \\ a' & \text{if } a \neq a' \end{cases}$ satisfies (FCB), so we get \bar{f} as

above and hence $i : [\mathbb{A}]\mathbb{A} \rightarrow \mathbb{A} + \mathbf{1}$ with

$$i(\langle a \rangle a') = \begin{cases} 0 & \text{if } a = a' \\ a' & \text{if } a \neq a' \end{cases}$$

It's not hard to see that i is both injective and surjective, hence an isomorphism in **Nom**. (Exercise)

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Proof. Just have to show that if $a, a' \# x$ then

$$(a, y) \sim (a', y') \Rightarrow f(x, a, y) = f(x, a', y')$$

so that $f(x, -, -)$ induces a function on equivalence classes.
(Equivariance of \bar{f} is automatic.)

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Proof. Just have to show that if $a, a' \# x$ then

$$(a, y) \sim (a', y') \Rightarrow f(x, a, y) = f(x, a', y')$$

But if $(a, y) \sim (a', y')$, then we can find $a'' \# (a, y, a', y', x)$ with $(a'' a) \cdot y = (a'' a') \cdot y'$. So

$$\begin{aligned} f(x, a, y) &= (a'' a) \cdot f(x, a, y) && \text{since } a, a'' \# f(x, a, y) \\ &= f(x, a'', (a'' a) \cdot y) && \text{since } a, a'' \# x \\ &= f(x, a'', (a'' a') \cdot y') \\ &= \dots \\ &= f(x, a', y') \end{aligned} \quad \square$$

Some properties of $[A](-)$

$$[A](X_1 \times \cdots \times X_n) \cong ([A]X_1) \times \cdots \times ([A]X_n)$$

$$[A](X_1 + \cdots + X_n) \cong ([A]X_1) + \cdots + ([A]X_n)$$

$$S \text{ discrete} \Rightarrow [A]S \cong S$$

$$[A](X \rightarrow_{\text{fs}} Y) \cong ([A]X) \rightarrow_{\text{fs}} ([A]Y) \quad (!!)$$