# Lecture 2: support

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  - each $\pi$ is a bijection $\mathbb{A} \cong \mathbb{A}$ (injective and surjective function)
  - group: multiplication is composition of functions $\pi' \circ \pi$; identity is identity function $\iota$; inverses are inverse functions $\pi^{-1}$.
  - $\pi$ finite means: $\{a \in \mathbb{A} \mid \pi(a) \neq a\}$ is finite.

Quiz: if $\mathbb{A} = \{0, 1, 2, 3, \ldots\}$, are these maps in $\mathbf{Perm}\,\mathbb{A}$?

- $0 \mapsto 1 \mapsto 0, k \mapsto k$ (for all $k \geq 2$)
- $2k \mapsto 2k + 1 \mapsto 2k$ (for all $k \geq 0$)
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- $2k \mapsto 2k + 1 \mapsto 2k$ (for all $k \geq 0$) is not finite
- $0 \mapsto 1 \mapsto 2 \mapsto 3 \mapsto \cdots$ is not a bijection

# Transposition

$(a\ b) \in \mathbf{Perm}\,\mathbb{A}$ is the function mapping $a$ to $b$, $b$ to $a$ and fixing all other names.

**Theorem:** every $\pi \in \mathbf{Perm}\,\mathbb{A}$ is equal to

$$(a_1\ b_1) \circ \cdots \circ (a_n\ b_n)$$

for some $a_i$ & $b_i$ (with $\pi\, a_i \neq a_i \neq b_i \neq \pi\, b_i$).

**Proof**...

by induction on the size of $\{a \mid \pi a \neq a\}$

– See [NSB, Theorem 1.15].

# Actions

An $\mathbf{Perm}\,\mathbb{A}$-action on a set $X$ is a function

$$(-)\cdot(-) : \mathbf{Perm}\,\mathbb{A} \times X \to X$$

satisfying for all $\pi, \pi' \in \mathbf{Perm}\,\mathbb{A}$ and $x \in X$

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NON-EXAMPLE : $a \mapsto \pi a$ does not give an action on $\{a, b\}$ – why?

# Running example

Action of **Perm** $\mathbb{A}$ on set of ASTs for $\lambda$-terms

$$Tr \triangleq \{t ::= \mathtt{V}\,a \mid \mathtt{A}(t,t) \mid \mathtt{L}(a,t)\}$$

$$\begin{aligned}
\pi \cdot \mathtt{V}\,a &= \mathtt{V}(\pi\, a) \\
\pi \cdot \mathtt{A}(t,t') &= \mathtt{A}(\pi \cdot t, \pi \cdot t') \\
\pi \cdot \mathtt{L}(a,t) &= \mathtt{L}(\pi\, a, \pi \cdot t)
\end{aligned}$$

This respects $\alpha$-equivalence $=_\alpha$ [lecture 1]

$$t =_\alpha t' \;\Rightarrow\; \pi \cdot t =_\alpha \pi \cdot t' \quad \text{(Exercise)}$$

and so induces an action on set of $\lambda$-terms
$\Lambda \triangleq \{[t]_\alpha \mid t \in Tr\}$:

$$\pi \cdot [t]_\alpha = [\pi \cdot t]_\alpha$$

# Nominal sets

are sets $X$ with with a $\mathbf{Perm}\,\mathbb{A}$-action satisfying

**Finite support property**: for each $x \in X$, there is a finite subset $A \subseteq \mathbb{A}$ that supports $x$, in the sense that for all $\pi \in \mathbf{Perm}\,\mathbb{A}$

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E.g. $\mathbb{A}$ is a nominal set—each $a \in \mathbb{A}$ is supported by any finite $A$ containing $a$.

$Tr$ and $\Lambda$ are nominal sets—any finite $A$ containing all the variables occurring (free, binding, or bound) in $t \in Tr$ supports $t$ and (hence) $[t]_\alpha$.

# Discrete nominal set

determined by a set $S$ has permutation action

$$\pi \cdot x \triangleq x \qquad (x \in S)$$

Each $x \in S$ is supported by $\emptyset$.

Trivial, but useful: ‘ordinary’ sets are included in nominal sets.

# Support of sets of names

$\mathbf{Perm}\,\mathbb{A}$ acts of sets of names $S \subseteq \mathbb{A}$ pointwise:
$\pi \cdot S \triangleq \{\pi\, a \mid a \in S\}$.

What is a support for the following sets of names?

- $S_1 \triangleq \{a\}$
- $S_2 \triangleq \mathbb{A} - \{a\}$
- $S_3 \triangleq \{a_0, a_2, a_4, \ldots\}$, supposing $\mathbb{A} = \{a_0, a_1, a_2, \ldots\}$

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  Answer: $\{a_0, a_2, a_4, \ldots\}$ is a support, and so is $\{a_1, a_3, a_5, \ldots\}$—but there is no finite support. $S_3$ does not exist in the 'world of nominal sets' (in that world $\mathbb{A}$ is infinite, but not enumerable—see later).

**Lemma.** In any nominal set $X$, every $x \in X$ possesses a smallest (wrt $\subseteq$) finite support, written $supp\, x$.

**Proof.** Suffices to show that if finite subsets $A_1$ and $A_2$ support $x$, then so does $A_1 \cap A_2$.

**Lemma.** In any nominal set $X$, every $x \in X$ possesses a smallest (wrt $\subseteq$) finite support, written $supp\, x$.

**Proof.** Suppose finite subsets $A_1$ and $A_2$ support $x$.

For any $a, a' \in \mathbb{A} - (A_1 \cap A_2)$ with $a \neq a'$,
claim that $(a\ a') \cdot x = x$.

Since every $\pi$ fixing the elements of $A_1 \cap A_2$ can be expressed as a composition of transpositions $(a\ a')$ with $a, a' \in \mathbb{A} - (A_1 \cap A_2)$ and $a \neq a'$, we have $\pi \cdot x = x$. □

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**Proof.** Suppose finite subsets $A_1$ and $A_2$ support $x$.

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claim that $(a\ a') \cdot x = x$.

Pick any $a'' \in \mathbb{A} - (A_1 \cup A_2 \cup \{a, a'\})$ (infinite, hence non-empty). Note that

$$(a, a'' \notin A_1) \vee (a, a'' \notin A_2)$$

so $(a\ a'') \cdot x = x$. Similarly $(a'\ a'') \cdot x = x$.

But $(a\ a') = (a\ a'') \circ (a'\ a'') \circ (a\ a'')$.

So $(a\ a') \cdot x = x$. $\Box$claim

# Free variables via support

Recall that $\Lambda = \{[t]_\alpha \mid t \in Tr\}$ has a **Perm** $\mathbb{A}$-action satisfying $\pi \cdot [t]_\alpha = [\pi \cdot t]_\alpha$

**Fact:** for any $[t]_\alpha \in \Lambda$, $supp([t]_\alpha)$ is the finite set $\mathbf{fv}\, t$ of free variables of any representative AST $t$.

$$\begin{aligned} \mathbf{fv}(\mathtt{V}\, a) &= \{a\} \\ \mathbf{fv}(\mathtt{A}(t, t')) &= (\mathbf{fv}\, t) \cup (\mathbf{fv}\, t') \\ \mathbf{fv}(\mathtt{L}(a, t)) &= (\mathbf{fv}\, t) - \{a\} \end{aligned}$$

(Exercise)

(Ex.3 on sheet)

Homework: Exercise sheet 1-3.