L11: Algebraic Path Problems with applications to Internet Routing
Lectures 12, 13

Timothy G. Griffin

timothy.griffin@cl.cam.ac.uk
Computer Laboratory
University of Cambridge, UK

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Algebra of Monoid Endomorphisms (AME) (See Gondran and Minoux 2008)

Let $(S, \oplus, \overline{0})$ be a commutative monoid.

$(S, \oplus, F \subseteq S \rightarrow S, \overline{0})$ is an algebra of monoid endomorphisms (AME) if

- $\forall f \in F, \; f(\overline{0}) = \overline{0}$
- $\forall f \in F, \forall b, c \in S, \; f(b \oplus c) = f(b) \oplus f(c)$

I will declare these as optional

- $\forall f, g \in F, \; f \circ g \in F$ (closed)
- $\exists i \in F, \forall s \in S, \; i(s) = s$
- $\exists \omega \in F, \forall n \in N, \; \omega(n) = \hat{0}$

Note: as with semirings, we may have to drop some of these axioms in order to model Internet routing ...

So why do we want AMEs?

Each (closed with ω and i) AME can be viewed as a semiring of functions. Suppose $(S,\ \oplus,\ F,\ \overline{0})$ is an algebra of monoid endomorphisms. We can turn it into a semiring

$$\mathbb{F} = (F,\ \hat{\oplus},\ \circ,\ \omega,\ i)$$

where $(f\ \hat{\oplus}\ g)(a) = f(a) \oplus g(a)$ and $(f\ \circ\ g)(a) = f(g(a))$.

But functions are hard to work with....

- All algorithms need to check equality over elements of a semiring
- $f = g$ means $\forall a \in S,\ f(a) = g(a)$
- S can be very large, or infinite ....

Lexicographic product of AMEs

$$(S,\ \oplus_S,\ F) \vec{\times} (T,\ \oplus_T,\ G) = (S \times T,\ \oplus_S \vec{\times} \oplus_T,\ F \times G)$$

Theorem 11.3

$$\mathrm{D}(S \vec{\times} T) \iff \mathrm{D}(S) \wedge \mathrm{D}(T) \wedge (\mathrm{C}(S) \vee \mathrm{K}(T))$$

Where

Property	Definition
D	$\forall a, b, f,\ f(a \oplus b) = f(a) \oplus f(b)$
C	$\forall a, b, f,\ f(a) = f(b) \implies a = b$
K	$\forall a, b, f,\ f(a) = f(b)$

Functional Union of AMEs

$$(S,\ \oplus,\ F) +_{\mathrm{m}} (S,\ \oplus,\ G) = (S,\ \oplus,\ F + G)$$

Fact

$$\mathrm{D}(S +_{\mathrm{m}} T) \iff \mathrm{D}(S) \wedge \mathrm{D}(T)$$

Where

Property	Definition
D	$\forall a, b, f : f(a \oplus b) = f(a) \oplus f(b)$

Left and Right

right

$$\mathbf{right}(S, \oplus, F) = (S, \oplus, \{i\})$$

left

$$\mathbf{left}(S, \oplus, F) = (S, \oplus, K(S))$$

where $K(S)$ represents all constant functions over S. For $a \in S$, define the function $\kappa_a(b) = a$. Then $K(S) = \{\kappa_a \mid a \in S\}$.

Facts

The following are always true.

$$\begin{array}{ll} \mathrm{D}(\mathbf{right}(S)) & \\ \mathrm{D}(\mathbf{left}(S)) & \text{(assuming } \oplus \text{ is idempotent)} \\ \mathrm{C}(\mathbf{right}(S)) & \\ \mathrm{K}(\mathbf{left}(S)) & \end{array}$$

Scoped Product (Think iBGP/eBGP)

$$S \Theta T = (S \vec{\times} \textbf{left}(T)) +_{\mathrm{m}} (\textbf{right}(S) \vec{\times} T)$$

Theorem 11.2

$$\mathrm{D}(S \Theta T) \iff \mathrm{D}(S) \wedge \mathrm{D}(T).$$

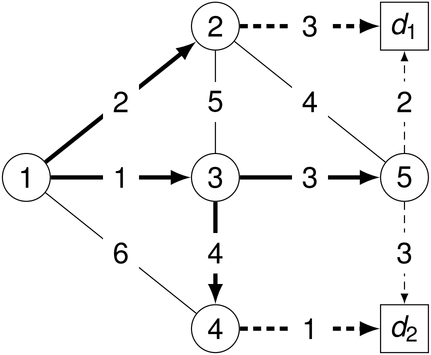
$$\begin{aligned}
& \mathrm{D}(S \Theta T) \\
& \mathrm{D}((S \vec{\times} \textbf{left}(T)) +_{\mathrm{m}} (\textbf{right}(S) \vec{\times} T)) \\
\iff & \mathrm{D}(S \vec{\times} \textbf{left}(T)) \wedge \mathrm{D}(\textbf{right}(S) \vec{\times} T) \\
\iff & \mathrm{D}(S) \wedge \mathrm{D}(\textbf{left}(T)) \wedge (\mathrm{C}(S) \vee \mathrm{K}(\textbf{left}(T))) \\
& \quad \wedge \mathrm{D}(\textbf{right}(S)) \wedge \mathrm{D}(T) \wedge (\mathrm{C}(\textbf{right}(S)) \vee \mathrm{K}(T)) \\
\iff & \mathrm{D}(S) \wedge \mathrm{D}(T)
\end{aligned}$$

Lexicographic Products in Metarouting. Alexander Gurney, Timothy G. Griffin. International Conference on Network Protocols (ICNP), 2007.

Routing Matrix vs. Path Matrix

- Inspired by the the Locator/ID split work
  - See Locator/ID Separation Protocol (LISP)
- Let's make a distinction between infrastructure nodes V and destinations D.
- Assume $V \cap D = \{\}$
- $\mathbf{M}$ is a $V \times D$ mapping matrix
  - $\mathbf{M}(v, d) \neq \infty$ means that destination (identifier) d is somehow attached to node (locator) v

Example of routing = path finding + mapping



$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} \infty & \infty \\ 3 & \infty \\ \infty & \infty \\ \infty & 1 \\ 2 & 3 \end{bmatrix} \end{array}$$

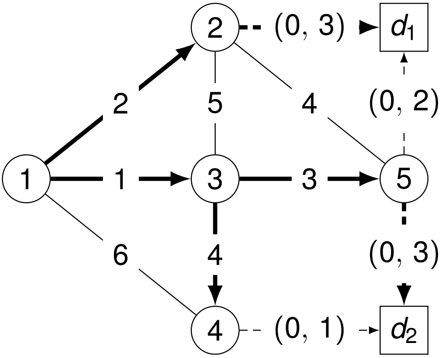
Mapping matrix

matrix	solves
$\mathbf{A}^*$	$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}$
$\mathbf{A}^*\mathbf{M}$	$\mathbf{F} = (\mathbf{A} \otimes \mathbf{F}) \oplus \mathbf{M}$

$$\mathbf{F} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} 5 & 6 \\ 3 & 7 \\ 5 & 5 \\ 9 & 1 \\ 2 & 3 \end{bmatrix} \end{array}$$

Routing matrix (paths implicit)

More Interesting Example : “Hot-Potato” Idiom — find attachment that is closest



$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} \infty & \infty \\ (0,3) & \infty \\ \infty & \infty \\ \infty & (0,1) \\ (0,2) & (0,3) \end{bmatrix} \end{array}$$

Mapping matrix

$$\mathbf{F} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} (2,3) & (4,3) \\ (0,3) & (4,3) \\ (3,2) & (3,3) \\ (7,2) & (0,1) \\ (0,2) & (0,3) \end{bmatrix} \end{array}$$

Routing matrix

General Case

$G = (V, E)$, n is the size of V.

A $n \times n$ (left) path matrix $\mathbf{L}$ solves an equation of the form

$$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I},$$

over semiring S.

D is a set of destinations, with size d.

A $n \times d$ routing matrix is defined as

$$\mathbf{F} = \mathbf{L} \triangleright \mathbf{M},$$

over some structure $(N, \Box, \triangleright)$, where $\triangleright \in S \to (N \to N)$.

routing $=$ path finding $+$ mapping

Does this make sense?

$$\mathbf{F}(i,\ d) = (\mathbf{L} \rhd \mathbf{M})(i,\ d) = \square_{q \in V} \mathbf{L}(i,\ q) \rhd \mathbf{M}(q,\ d).$$

- Once again we are leaving paths implicit in the construction.
- Routing paths are best paths to egress nodes, selected with respect to $\square$-minimality.
- $\square$-minimality can be very different from selection involved in path finding.

When we are lucky ...

matrix	solves
$\mathbf{A}^*$	$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}$
$\mathbf{A}^* \rhd \mathbf{M}$	$\mathbf{F} = (\mathbf{A} \rhd \mathbf{F}) \Box \mathbf{M}$

When does this happen?

When $(N, \Box, \rhd)$ is a (left) semi-module over the semiring S[a].

[a] A model of Internet routing using semi-modules. John N. Billings and Timothy G. Griffin. RelMiCS11/AKA6 2009

(left) Semi-modules

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ is a semiring.

A (left) semi-module over S

Is a structure $(N, \Box, \rhd, \overline{0}_N)$, where

- $(N, \Box, \overline{0}_N)$ is a commutative monoid
- $\rhd$ is a function $\rhd \in (S \times N) \to N$
- $(a \otimes b) \rhd m = a \rhd (b \rhd m)$
- $\overline{0} \rhd m = \overline{0}_N$
- $s \rhd \overline{0}_N = \overline{0}_N$
- $\overline{1} \rhd m = m$

and distributivity holds,

$$
\begin{array}{lcrcl}
\mathrm{LD} & : & s \rhd (m \Box n) & = & (s \rhd m) \Box (s \rhd n) \\
\mathrm{RD} & : & (s \oplus t) \rhd m & = & (s \rhd m) \Box (t \rhd m)
\end{array}
$$

Example : Hot-Potato

S idempotent and selective

$$\begin{array}{rcl} S & = & (S,\ \oplus_S,\ \otimes_S) \\ T & = & (T,\ \oplus_T,\ \otimes_T) \\ \rhd_{\text{fst}} & \in & S \to (S \times T) \to (S \times T) \\ s_1 \rhd_{\text{fst}} (s_2,\ t) & = & (s_1 \otimes_S s_2,\ t) \end{array}$$

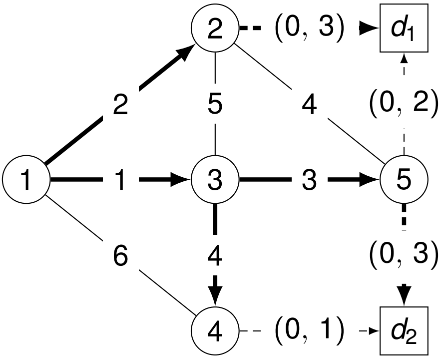
$$\text{Hot}(S,\ T) = (S \times T,\ \vec{\oplus},\ \rhd_{\text{fst}}),$$

where $\vec{\oplus}$ is the (left) lexicographic product of $\oplus_S$ and $\oplus_T$.

Define $\rhd_{\text{hp}}$ on matrices

$$(\mathbf{L} \rhd_{\text{hp}} \mathbf{M})(i,\ d) = \vec{\bigoplus}_{q \in V} \mathbf{L}(i,\ q) \rhd_{\text{fst}} \mathbf{M}(q,\ d)$$

Example of hot-potato routing



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Mapping matrix

$$\mathbf{F} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} (2,3) & (4,3) \\ (0,3) & (4,3) \\ (3,2) & (3,3) \\ (7,2) & (0,1) \\ (0,2) & (0,3) \end{bmatrix} \end{array}$$

Routing matrix

matrix	solves
$\mathbf{A}^*$	$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}$
$\mathbf{A}^* \rhd_{\mathrm{hp}} \mathbf{M}$	$\mathbf{F} = (\mathbf{A} \rhd_{\mathrm{hp}} \mathbf{F}) \vec{\oplus} \mathbf{M}$

Example : Cold-Potato

T idempotent and selective

$$\begin{array}{rcl} S & = & (S, \oplus_S, \otimes_S) \\ T & = & (T, \oplus_T, \otimes_T) \\ \rhd_{\mathrm{fst}} & \in & S \to (S \times T) \to (S \times T) \\ s_1 \rhd_{\mathrm{fst}} (s_2, t) & = & (s_1 \otimes_S s_2, t) \end{array}$$

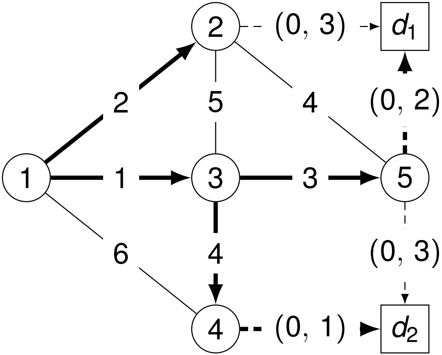
$$\mathrm{Cold}(S, T) = (S \times T, \overleftarrow{\oplus}, \rhd_{\mathrm{fst}}),$$

where $\vec{\oplus}$ is the (left) lexicographic product of $\oplus_S$ and $\oplus_T$.

Define $\rhd_{\mathrm{cp}}$ on matrices

$$(\mathbf{L} \rhd_{\mathrm{cp}} \mathbf{M})(i, d) = \overleftarrow{\bigoplus}_{q \in V} \mathbf{L}(i, q) \rhd_{\mathrm{fst}} \mathbf{M}(q, d)$$

Example of cold-potato routing



$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \left[\begin{array}{cc} \infty & \infty \\ (0,\ 3) & \infty \\ \infty & \infty \\ \infty & (0,\ 1) \\ (0,\ 2) & (0,\ 3) \end{array} \right] \end{array}$$

Mapping matrix

$$\mathbf{F} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \left[\begin{array}{cc} (4,\ 2) & (5,\ 1) \\ (4,\ 2) & (9,\ 1) \\ (3,\ 2) & (4,\ 1) \\ (7,\ 2) & (0,\ 1) \\ (0,\ 2) & (7,\ 1) \end{array} \right] \end{array}$$

Routing matrix

matrix	solves
$\mathbf{A}^*$	$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}$
$\mathbf{A}^* \rhd_{\mathrm{cp}} \mathbf{M}$	$\mathbf{F} = \mathbf{A} \rhd_{\mathrm{cp}} \mathbf{F} \overleftarrow{\oplus} \mathbf{M}$