L11: Algebraic Path Problems with applications to Internet Routing
Lectures 05—07

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An observation concerning 0-stable semirings

Suppose that p_1 is a path from i to k, p_2 is a path from k to k (a loop), and p_3 is a path from k to j.

Claim

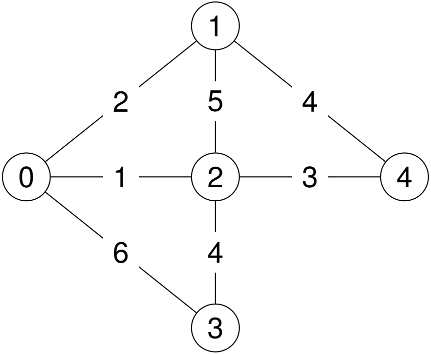
If the graph is weighted over a 0-stable semiring ($\overline{1} \oplus a = a \oplus \overline{1} = \overline{1}$), then

$$w(p_1 p_3) \leq_{\oplus}^{L} w(p_1 p_2 p_3).$$

In other words, for such semirings it does not pay to go around loops seeking a minimum path weight.

$$\begin{array}{lcl}
w(p_1 p_3) \oplus w(p_1 p_2 p_3) & = & (w(p_1) \otimes w(p_3)) \oplus (w(p_1) \otimes w(p_2) \otimes w(p_3)) \\
 & = & w(p_1) \otimes (\overline{1} \oplus w(p_2)) \otimes w(p_3) \\
 & = & w(p_1) \otimes \overline{1} \otimes w(p_3) \\
 & = & w(p_1) \otimes w(p_3) \\
 & = & w(p_1 p_3)
\end{array}$$

Shortest paths example, $(\mathbb{N}^{\infty}, \min, +)$



The adjacency matrix

$$\mathbf{A} \quad = \quad \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \left[\begin{array}{ccccc} \infty & 2 & 1 & 6 & \infty \\ 2 & \infty & 5 & \infty & 4 \\ 1 & 5 & \infty & 4 & 3 \\ 6 & \infty & 4 & \infty & \infty \\ \infty & 4 & 3 & \infty & \infty \end{array}\right] \end{array}$$

Note that the longest *shortest path* is $(1,\ 0,\ 2,\ 3)$ of length 3 and weight 7.

$(\min, +)$ example

Our theorem tells us that $\mathbf{A}^* = \mathbf{A}^{(n-1)} = \mathbf{A}^{(4)}$

$$\mathbf{A}^* = \mathbf{A}^{(4)} = \mathbf{I} \ \min \ \mathbf{A} \ \min \ \mathbf{A}^2 \ \min \ \mathbf{A}^3 \ \min \ \mathbf{A}^4 = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \left[\begin{array}{ccccc} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{array} \right] \end{array}$$

(min, +) example

$$\mathbf{A} = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \infty & \underline{2} & \underline{1} & 6 & \infty \\ \underline{2} & \infty & 5 & \infty & \underline{4} \\ \underline{1} & 5 & \infty & \underline{4} & \underline{3} \\ 6 & \infty & \underline{4} & \infty & \infty \\ \infty & \underline{4} & \underline{3} & \infty & \infty \end{bmatrix} \end{array} \quad \mathbf{A}^3 = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 8 & 4 & 3 & 8 & 10 \\ 4 & 8 & 7 & \underline{7} & 6 \\ 3 & 7 & 8 & 6 & 5 \\ 8 & \underline{7} & 6 & 11 & 10 \\ 10 & 6 & 5 & 10 & 12 \end{bmatrix} \end{array}$$

$$\mathbf{A}^2 = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 2 & 6 & 7 & \underline{5} & \underline{4} \\ 6 & 4 & \underline{3} & 8 & 8 \\ 7 & \underline{3} & 2 & 7 & 9 \\ \underline{5} & 8 & 7 & 8 & \underline{7} \\ \underline{4} & 8 & 9 & \underline{7} & 6 \end{bmatrix} \end{array} \quad \mathbf{A}^4 = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 4 & 8 & 9 & 7 & 6 \\ 8 & 6 & 5 & 10 & 10 \\ 9 & 5 & 4 & 9 & 11 \\ 7 & 10 & 9 & 10 & 9 \\ 6 & 10 & 11 & 9 & 8 \end{bmatrix} \end{array}$$

First appearance of final value is in red and underlined. Remember: we are looking at all paths of a given length, even those with cycles!

A "better" way — our basic algorithm

$$\begin{array}{rcl} \mathbf{A}^{\langle 0 \rangle} & = & \mathbf{I} \\ \mathbf{A}^{\langle k+1 \rangle} & = & \mathbf{A}\mathbf{A}^{\langle k \rangle} \oplus \mathbf{I} \end{array}$$

Lemma

$$\mathbf{A}^{\langle k \rangle} = \mathbf{A}^{(k)} = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k$$

back to $(\min, +)$ example

$$\mathbf{A}^{\langle 1 \rangle} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 6 & \infty \\ 2 & 0 & 5 & \infty & 4 \\ 1 & 5 & 0 & 4 & 3 \\ 6 & \infty & 4 & 0 & \infty \\ \infty & 4 & 3 & \infty & 0 \end{bmatrix} \end{array}$$

$$\mathbf{A}^{\langle 3 \rangle} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{A}^{\langle 2 \rangle} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 8 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 8 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{bmatrix} \end{array}$$

A note on $\mathbf{A}$ vs. $\mathbf{A} \oplus \mathbf{I}$

Lemma 6.0

If $\oplus$ is idempotent, then

$$(\mathbf{A} \oplus \mathbf{I})^k = \mathbf{A}^{(k)}.$$

Proof. Base case: When $k = 0$ both expressions are $\mathbf{I}$.
Assume $(\mathbf{A} \oplus \mathbf{I})^k = \mathbf{A}^{(k)}$. Then

$$\begin{array}{rcl}
(\mathbf{A} \oplus \mathbf{I})^{k+1} & = & (\mathbf{A} \oplus \mathbf{I})(\mathbf{A} \oplus \mathbf{I})^k \\
 & = & (\mathbf{A} \oplus \mathbf{I})\mathbf{A}^{(k)} \\
 & = & \mathbf{A}\mathbf{A}^{(k)} \oplus \mathbf{A}^{(k)} \\
 & = & \mathbf{A}(\mathbf{I} \oplus \mathbf{A} \oplus \cdots \oplus \mathbf{A}^k) \oplus \mathbf{A}^{(k)} \\
 & = & \mathbf{A} \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^{k+1} \oplus \mathbf{A}^{(k)} \\
 & = & \mathbf{A}^{k+1} \oplus \mathbf{A}^{(k)} \\
 & = & \mathbf{A}^{(k+1)}
\end{array}$$

Solving (some) equations

Theorem 6.1

If $\mathbf{A}$ is q-stable, then $\mathbf{X} = \mathbf{A}^*$ solves the equations

$$\mathbf{X} = \mathbf{A}\mathbf{X} \oplus \mathbf{I}$$

and

$$\mathbf{X} = \mathbf{X}\mathbf{A} \oplus \mathbf{I}.$$

For example,

$$\begin{array}{rcl}
\mathbf{A}^* & = & \mathbf{A}^{(q)} \\
& = & \mathbf{A}^{(q+1)} \\
& = & \mathbf{A}^{q+1} \oplus \mathbf{A}^q \oplus \ldots \oplus \mathbf{A}^2 \oplus \mathbf{A} \oplus \mathbf{I} \\
& = & \mathbf{A}(\mathbf{A}^q \oplus \mathbf{A}^{q-1} \oplus \ldots \oplus \mathbf{A} \oplus \mathbf{I}) \oplus \mathbf{I} \\
& = & \mathbf{A}\mathbf{A}^{(q)} \oplus \mathbf{I} \\
& = & \mathbf{A}\mathbf{A}^* \oplus \mathbf{I}
\end{array}$$

Note that if we replace the assumption "$\mathbf{A}$ is q-stable" with "$\mathbf{A}^*$ exists," then we require that $\otimes$ distributes over infinite sums.

A more general result

Theorem Left-Right

If $\mathbf{A}$ is q-stable, then $\mathbf{X} = \mathbf{A}^*\mathbf{B}$ solves the equations

$$\mathbf{X} = \mathbf{A}\mathbf{X} \oplus \mathbf{B}$$

and $\mathbf{X} = \mathbf{B}\mathbf{A}^*$ solves

$$\mathbf{X} = \mathbf{X}\mathbf{A} \oplus \mathbf{B}.$$

For example,

$$\begin{aligned}
\mathbf{A}^*\mathbf{B} &= \mathbf{A}^{(q)}\mathbf{B} \\
&= \mathbf{A}^{(q+1)}\mathbf{B} \\
&= (\mathbf{A}^{q+1} \oplus \mathbf{A}^{q} \oplus \ldots \oplus \mathbf{A}^{2} \oplus \mathbf{A} \oplus \mathbf{I})\mathbf{B} \\
&= (\mathbf{A}^{q+1} \oplus \mathbf{A}^{q} \oplus \ldots \oplus \mathbf{A}^{2} \oplus \mathbf{A})\mathbf{B} \oplus \mathbf{B} \\
&= \mathbf{A}(\mathbf{A}^{q} \oplus \mathbf{A}^{q-1} \oplus \ldots \oplus \mathbf{A} \oplus \mathbf{I})\mathbf{B} \oplus \mathbf{B} \\
&= \mathbf{A}(\mathbf{A}^{(q)}\mathbf{B}) \oplus \mathbf{B} \\
&= \mathbf{A}(\mathbf{A}^*\mathbf{B}) \oplus \mathbf{B}
\end{aligned}$$

Use Theorem Left-Right to Work this out

Theorem (John Conway, 1971)

If

$$\mathbf{A} = \left(\begin{array}{c|c} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \hline \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{array}\right)$$

then $\mathbf{A}^*$ can be written as

$$\left(\begin{array}{c|c} (\mathbf{A}_{1,1} \oplus \mathbf{A}_{1,2}\mathbf{A}_{2,2}^*\mathbf{A}_{2,1})^* & \mathbf{A}_{1,1}^*\mathbf{A}_{1,2}(\mathbf{A}_{2,2} \oplus \mathbf{A}_{2,1}\mathbf{A}_{1,1}^*\mathbf{A}_{1,2})^* \\ \hline \mathbf{A}_{2,2}^*\mathbf{A}_{2,1}(\mathbf{A}_{1,1} \oplus \mathbf{A}_{1,2}\mathbf{A}_{2,2}^*\mathbf{A}_{2,1})^* & (\mathbf{A}_{2,2} \oplus \mathbf{A}_{2,1}\mathbf{A}_{1,1}^*\mathbf{A}_{1,2})^* \end{array}\right)$$

The "best" solution

Suppose $\mathbf{Y}$ is a matrix such that

$$\mathbf{Y} = \mathbf{AY} \oplus \mathbf{I}$$

$$\begin{aligned}
\mathbf{Y} &= \mathbf{AY} \oplus \mathbf{I} \\
&= \mathbf{A}^1\mathbf{Y} \oplus \mathbf{A}^{(0)} \\
&= \mathbf{A}((\mathbf{AY} \oplus \mathbf{I})) \oplus \mathbf{I} \\
&= \mathbf{A}^2\mathbf{Y} \oplus \mathbf{A} \oplus \mathbf{I} \\
&= \mathbf{A}^2\mathbf{Y} \oplus \mathbf{A}^{(1)} \\
&\vdots \quad \vdots \quad \vdots \\
&= \mathbf{A}^{k+1}\mathbf{Y} \oplus \mathbf{A}^{(k)}
\end{aligned}$$

If $\mathbf{A}$ is q-stable and $q < k$, then

$$\mathbf{Y} = \mathbf{A}^k\mathbf{Y} \oplus \mathbf{A}^*$$

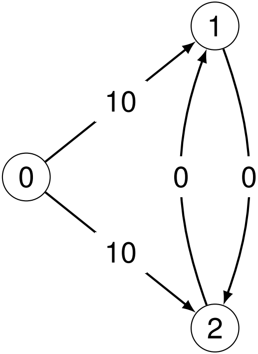
$$\mathbf{Y} \trianglelefteq^L_\oplus \mathbf{A}^*$$

and if $\oplus$ is idempotent, then

$$\mathbf{Y} \leq^L_\oplus \mathbf{A}^*$$

So $\mathbf{A}^*$ is the largest solution. What does this mean in terms of the sp semiring?

Example with zero weighted cycles using sp semiring



$$\mathbf{A} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left[\begin{array}{ccc} \infty & 10 & 10 \\ \infty & \infty & 0 \\ \infty & 0 & \infty \end{array}\right] \end{array}$$

$\mathbf{A}^*$ ($= \mathbf{A} \oplus \mathbf{I}$ in this case) solves

$$\mathbf{X} = \mathbf{X}\mathbf{A} \oplus \mathbf{I}.$$

But so does this (dishonest) matrix!

$$\mathbf{F} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left[\begin{array}{ccc} 0 & 9 & 9 \\ \infty & 0 & 0 \\ \infty & 0 & 0 \end{array}\right] \end{array}$$

For example :

$$\begin{aligned} & (\mathbf{FA})(0, 1) \\ = \; & \min_{q \in \{0,1,2\}} \mathbf{F}(0, q) + \mathbf{A}(q, 1) \\ = \; & \min(0 + 10, 9 + \infty, 9 + 0) \\ = \; & 9 \\ = \; & \mathbf{F}(0, 1) \end{aligned}$$

Recall our basic iterative algorithm

$$\begin{aligned} \mathbf{A}^{\langle 0 \rangle} &= \mathbf{I} \\ \mathbf{A}^{\langle k+1 \rangle} &= \mathbf{A}\mathbf{A}^{\langle k \rangle} \oplus \mathbf{I} \end{aligned}$$

A closer look ...

$$\begin{aligned} \mathbf{A}^{\langle k+1 \rangle}(i,j) &= \mathbf{I}(i,j) \oplus \bigoplus_{u} \mathbf{A}(i,u)\mathbf{A}^{\langle k \rangle}(u,j) \\ &= \mathbf{I}(i,j) \oplus \bigoplus_{(i,u) \in E} \mathbf{A}(i,u)\mathbf{A}^{\langle k \rangle}(u,j) \end{aligned}$$

This is the basis of distributed Bellman-Ford algorithms — a node i computes routes to a destination j by applying its link weights to the routes learned from its immediate neighbors. It then makes these routes available to its neighbors and the process continues...

What if we start iteration in an arbitrary state $\mathbf{M}$?

In a distributed environment the topology (captured here by **A**) can change and the state of the computation can start in an arbitrary state (with respect to a new **A**).

$$
\begin{aligned}
\mathbf{A}_{\mathbf{M}}^{\langle 0 \rangle} &= \mathbf{M} \\
\mathbf{A}_{\mathbf{M}}^{\langle k+1 \rangle} &= \mathbf{A}\mathbf{A}_{\mathbf{M}}^{\langle k \rangle} \oplus \mathbf{I}
\end{aligned}
$$

Lemma 6.4

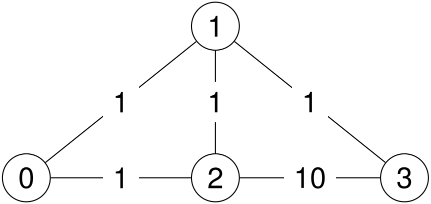
For $1 \leq k$,

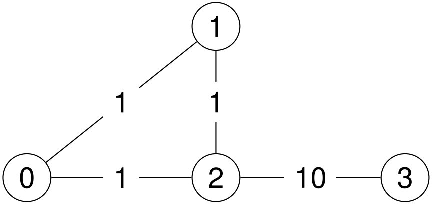
$$\mathbf{A}_{\mathbf{M}}^{\langle k \rangle} = \mathbf{A}^k\mathbf{M} \oplus \mathbf{A}^{(k-1)}$$

If **A** is q-stable and $q < k$, then

$$\mathbf{A}_{\mathbf{M}}^{\langle k \rangle} = \mathbf{A}^k\mathbf{M} \oplus \mathbf{A}^*$$

RIP-like example — counting to convergence (1)





Adjacency matrix $\mathbf{A}_1$

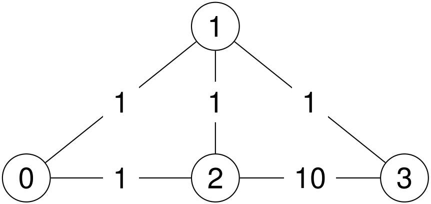
$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \infty & 1 & 1 & \infty \\ 1 & \infty & 1 & 1 \\ 1 & 1 & \infty & 10 \\ \infty & 1 & 10 & \infty \end{bmatrix} \end{array}$$

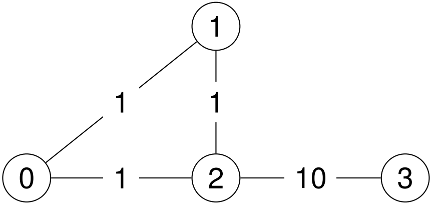
Adjacency matrix $\mathbf{A}_2$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \infty & 1 & 1 & \infty \\ 1 & \infty & 1 & \infty \\ 1 & 1 & \infty & 10 \\ \infty & \infty & 10 & \infty \end{bmatrix} \end{array}$$

See RFC 1058.

RIP-like example — counting to convergence (2)





The solution $\mathbf{A}_1^*$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{array}\right] \end{array}$$

The solution $\mathbf{A}_2^*$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 1 & 11 \\ 1 & 0 & 1 & 11 \\ 1 & 1 & 0 & 10 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array}$$

RIP-like example — counting to convergence (3)

The scenario: we arrived at $\mathbf{A}_1^*$, but then links $\{(1,3),\ (3,1)\}$ fail. So we start iterating using the new matrix $\mathbf{A}_2$.

Let $\mathbf{B}_K$ represent $\mathbf{A}_{2}{}_{\mathbf{M}}^{\langle k \rangle}$, where $\mathbf{M} = \mathbf{A}_1^*$.

RIP-like example — counting to convergence (4)

$$\mathbf{B}_0 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{array}\right] \end{array} \qquad \mathbf{B}_3 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 0 & 4 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array}$$

$$\mathbf{B}_1 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 2 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array} \qquad \mathbf{B}_4 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 5 \\ 1 & 0 & 1 & 5 \\ 1 & 1 & 0 & 5 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array}$$

$$\mathbf{B}_2 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array} \qquad \mathbf{B}_5 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 6 \\ 1 & 0 & 1 & 6 \\ 1 & 1 & 0 & 6 \\ 11 & 11 & 10 & 0 \end{array}\right] \end{array}$$

RIP-like example — counting to convergence (5)

$$\mathbf{B}_6 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 7 \\ 1 & 0 & 1 & 7 \\ 1 & 1 & 0 & 7 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

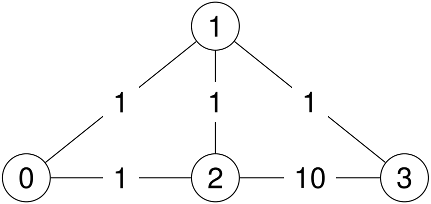
$$\mathbf{B}_7 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 8 \\ 1 & 0 & 1 & 8 \\ 1 & 1 & 0 & 8 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

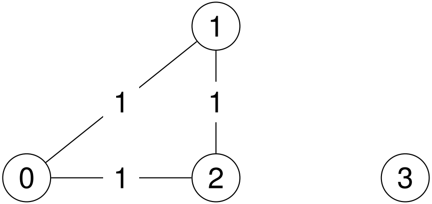
$$\mathbf{B}_8 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 9 \\ 1 & 0 & 1 & 9 \\ 1 & 1 & 0 & 9 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_9 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 10 \\ 1 & 0 & 1 & 10 \\ 1 & 1 & 0 & 10 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_{10} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 11 \\ 1 & 0 & 1 & 11 \\ 1 & 1 & 0 & 10 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

RIP-like example — counting to infinity (1)





The solution $\mathbf{A}_1^*$

$$\begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

The solution $\mathbf{A}_3^*$

$$\begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & \infty \\ 1 & 0 & 1 & \infty \\ 1 & 1 & 0 & \infty \\ \infty & \infty & \infty & 0 \end{bmatrix} \end{array}$$

Now let $\mathbf{B}_K$ represent $\mathbf{A}_{3\,\mathbf{M}}^{\langle k \rangle}$, where $\mathbf{M} = \mathbf{A}_1^*$.

RIP-like example — counting to infinity (2)

$$\mathbf{B}_0 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 \\ 3 & 2 & 1 & 2 & 0 \end{array}$$

$$\mathbf{B}_1 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 3 \\ 2 & 1 & 1 & 0 & 2 \\ 3 & \infty & \infty & \infty & 0 \end{array}$$

$$\mathbf{B}_2 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 3 \\ 1 & 1 & 0 & 1 & 3 \\ 2 & 1 & 1 & 0 & 3 \\ 3 & \infty & \infty & \infty & 0 \end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\mathbf{B}_{376} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 377 \\ 1 & 1 & 0 & 1 & 377 \\ 2 & 1 & 1 & 0 & 377 \\ 3 & \infty & \infty & \infty & 0 \end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

$$\mathbf{B}_{998} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 999 \\ 1 & 1 & 0 & 1 & 999 \\ 2 & 1 & 1 & 0 & 999 \\ 3 & \infty & \infty & \infty & 0 \end{array}$$

$$\vdots \quad \vdots \quad \vdots$$

RIP-like example — What's going on?

Recall

$$\mathbf{A}_{\mathbf{M}}^{\langle k \rangle}(i,\ j) = \mathbf{A}^k\mathbf{M}(i,\ j) \oplus \mathbf{A}^*(i,\ j)$$

- $\mathbf{A}^*$(i, j) may be arrived at very quickly
- but $\mathbf{A}^k\mathbf{M}(i,\ j)$ may be better until a very large value of k is reached (counting to convergence)
- or it may always be better (counting to infinity).

Solutions?

- RIP: $\infty = 16$
- We will explore various ways of adding paths to metrics and eliminating those paths with loops ....

Goal

$G = (V,\ E)$

A semiring S, such that if A is an adjaceny matrix over S with

$$A(i,j) = \begin{cases} \{(i,j)\} & \text{if } (i,j) \in E \\ \{\} & \text{otherwise} \end{cases}$$

then

$$A^*(i,j) = \text{the set of all elementary (no loops) paths from } i \text{ to } j.$$

We could attempt to directly define such an algebra. But instead we will build it step-by-step using simple constructions ...

Lifted Product

Lifted product semigroup

Assume $(S, \otimes)$ is a semigroup. Let $\mathrm{lift}_\times(S) \equiv (\mathcal{P}_{\mathrm{fin}}(S), \hat{\otimes})$ where

$$X \hat{\otimes} Y = \{x \otimes y \mid x \in X,\ y \in Y\}$$

, where $X, Y \in \mathcal{P}_{\mathrm{fin}}(S)$, the set of finite subsets of S.

Lifted semiring

If $\overline{1}$ is the identity for $\otimes$, then

$$\mathrm{lift}(S) = (\mathcal{P}_{\mathrm{fin}}(S),\ \cup,\ \hat{\otimes},\ \{\},\ \{\overline{1}\})$$

is a semiring. Note that $\{\}$ is an annihilator for $\hat{\otimes}$.

When does $\mathrm{lift}(S)$ have an annihilator for $\cup$?

Operation for inserting a zero

$$\mathrm{add\_zero}(\overline{0},\ (S,\ \oplus,\ \otimes)) = (S \uplus \{\overline{0}\}, \oplus_{\overline{0}},\ \otimes_{\overline{0}})$$

where

$$a \oplus_{\overline{0}} b \quad = \quad \begin{cases} a & (\text{if } b = \mathrm{inr}(\overline{0})) \\ b & (\text{if } a = \mathrm{inr}(\overline{0})) \\ \mathrm{inl}(x \oplus y) & (\text{if } a = \mathrm{inl}(x),\, b = \mathrm{inl}(y)) \end{cases}$$

$$a \otimes_{\overline{0}} b \quad = \quad \begin{cases} \mathrm{inr}(\overline{0}) & (\text{if } b = \mathrm{inr}(\overline{0})) \\ \mathrm{inr}(\overline{0}) & (\text{if } a = \mathrm{inr}(\overline{0})) \\ \mathrm{inl}(x \otimes y) & (\text{if } a = \mathrm{inl}(x),\, b = \mathrm{inl}(y)) \end{cases}$$

disjoint union

$$A \uplus B \equiv \{\mathrm{inl}(a) \mid a \in A\} \cup \{\mathrm{inr}(b) \mid b \in B\}$$

Operation for inserting a one

$$\text{add\_one}(\overline{1},\ (S,\ \oplus,\ \otimes)) = (S \uplus \{\overline{1}\}, \oplus_{\overline{1}},\ \otimes_{\overline{1}})$$

where

$$a \oplus_{\overline{1}} b \quad = \quad \begin{cases} \text{inr}(\overline{1}) & (\text{if } b = \text{inr}(\overline{1})) \\ \text{inr}(\overline{1}) & (\text{if } a = \text{inr}(\overline{1})) \\ \text{inl}(x \oplus y) & (\text{if } a = \text{inl}(x),\ b = \text{inl}(y)) \end{cases}$$

$$a \otimes_{\overline{1}} b \quad = \quad \begin{cases} a & (\text{if } b = \text{inr}(\overline{1})) \\ b & (\text{if } a = \text{inr}(\overline{1})) \\ \text{inl}(x \otimes y) & (\text{if } a = \text{inl}(x),\ b = \text{inl}(x)) \end{cases}$$

Reductions

If $(S, \oplus, \otimes)$ is a semiring and r is a function from S to S, then r is a reduction if for all a and b in S

1. $r(a) = r(r(a))$
2. $r(a \oplus b) = r(r(a) \oplus b) = r(a \oplus r(b))$
3. $r(a \otimes b) = r(r(a) \otimes b) = r(a \otimes r(b))$

Note that if either operation has an identity, then the first axioms is not needed. For example,

$$r(a) = r(a \oplus \overline{0}) = r(r(a) \oplus \overline{0}) = r(r(a))$$

Reduce operation

If $(S, \oplus, \otimes)$ is semiring and r is a reduction, then let $\mathrm{red}_r(S) = (S_r, \oplus_r, \otimes_r)$ where

1. $S_r = \{s \in S \mid r(s) = s\}$
2. $x \oplus_r y = r(x \oplus y)$
3. $x \otimes_r y = r(x \otimes y)$

Is the result always semiring?

Finally : A semiring of elementary paths

Semigroup of Sequences $\mathrm{seq}(X)$

- carrier : finite sequences over elements of X
- operation : concatenation
- identity : the empty string ϵ

Let X be a set of sequences over $\mathrm{lift}(\mathrm{seq}(E))$, and let

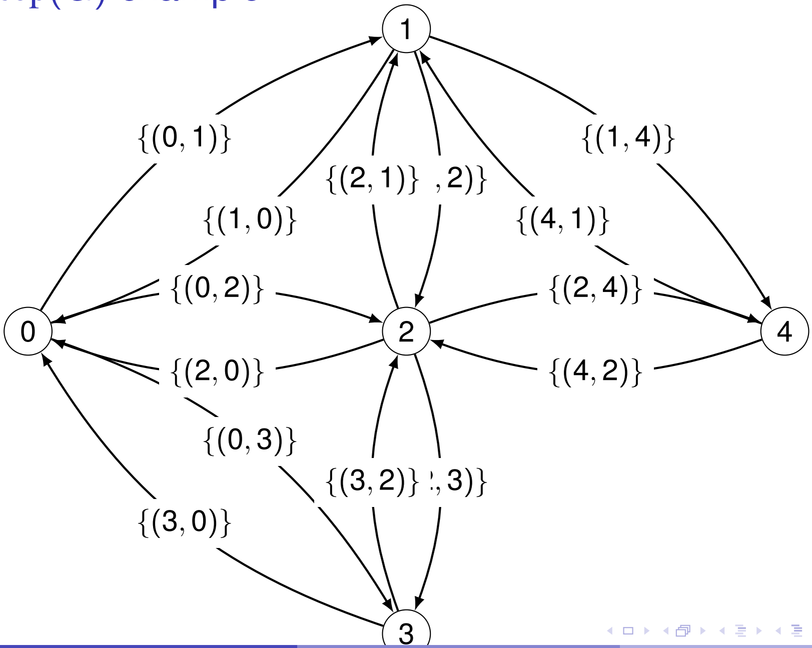
$$r(X) = \{p \in X \mid p \text{ is an elementary path in } G\}$$

Semiring of Elementary Paths

$$\mathrm{sep}(G) = \mathrm{red}_r(\mathrm{lift}(\mathrm{seq}(E)))$$

Preview of next problem set: In order to check that $\mathrm{sep}(G)$ is indeed a semiring, we only need understand the functions $\mathrm{lift}(\_)$ and $\mathrm{red}\_(\_)$.

sep(G) example



$\mathrm{sep}(G)$ example, adjacency matrix

$$\mathbf{I} = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \{\epsilon\} & \{\} & \{\} & \{\} & \{\} \\ \{\} & \{\epsilon\} & \{\} & \{\} & \{\} \\ \{\} & \{\} & \{\epsilon\} & \{\} & \{\} \\ \{\} & \{\} & \{\} & \{\epsilon\} & \{\} \\ \{\} & \{\} & \{\} & \{\} & \{\epsilon\} \end{bmatrix} \end{array}$$

$$\mathbf{A} = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \{\} & \{[(0,1)]\} & \{[(0,2)]\} & \{[(0,3)]\} & \{\} \\ \{[(1,0)]\} & \{\} & \{[(1,2)]\} & \{\} & \{[(1,4)]\} \\ \{[(2,0)]\} & \{[(2,1)]\} & \{\} & \{[(2,3)]\} & \{[(2,4)]\} \\ \{[(3,0)]\} & \{\} & \{[(3,2)]\} & \{\} & \{\} \\ \{\} & \{[(4,1)]\} & \{[(4,2)]\} & \{\} & \{\} \end{bmatrix} \end{array}$$

Here I write a non-empty path p as $[p]$.

sep(G) example, solution

$$
\begin{array}{rcl}
\mathbf{A}^*(0,0) & = & \{\epsilon\} \\
\mathbf{A}^*(0,4) & = & \left\{ \begin{array}{l}
[(0,1),(1,4)], \\
[(0,1),(1,2),(2,4)], \\
[(0,2),(2,4)], \\
[(0,2),(2,1),(1,4)], \\
[(0,3),(3,2),(2,4)], \\
[(0,3),(3,2),(2,1),(1,4)]
\end{array} \right\}
\end{array}
$$

Direct Product of Semigroups

Let $(S, \oplus_S)$ and $(T, \oplus_T)$ be semigroups.

Definition (Direct product semigroup)

The direct product is denoted $(S, \oplus_S) \times (T, \oplus_T) = (S \times T, \oplus)$, where $\oplus = \oplus_S \times \oplus_T$ is defined as

$$(s_1, t_1) \oplus (s_2, t_2) = (s_1 \oplus_S s_2, \; t_1 \oplus_T t_2).$$

Lexicographic Product of Semigroups

Definition (Lexicographic product semigroup)

Suppose that semigroup $(S, \oplus_S)$ is commutative, idempotent, and selective and that $(T, \oplus_T)$ is a semigroup. The lexicographic product is denoted $(S, \oplus_S) \vec{\times} (T, \oplus_T) = (S \times T, \vec{\oplus})$, where $\vec{\oplus} = \oplus_S \vec{\times} \oplus_T$ is defined as

$$(s_1, t_1) \vec{\oplus} (s_2, t_2) = \begin{cases} (s_1 \oplus_S s_2, t_1 \oplus_T t_2) & s_1 = s_1 \oplus_S s_2 = s_2 \\ (s_1 \oplus_S s_2, t_1) & s_1 = s_1 \oplus_S s_2 \neq s_2 \\ (s_1 \oplus_S s_2, t_2) & s_1 \neq s_1 \oplus_S s_2 = s_2 \end{cases}$$

Lexicographic product of Bi-semigroups

$$(S,\ \oplus_S,\ \otimes_S)\ \vec{\times}\ (T,\ \oplus_T,\ \otimes_T) = (S \times T,\ \oplus_S \vec{\times} \oplus_T,\ \otimes_S \times \otimes_T)$$

Theorem

If $\oplus_S$ is commutative, idempotent, and selective, then

$$\text{LD}(S \vec{\times} T) \iff \text{LD}(S) \wedge \text{LD}(T) \wedge (\text{LC}(S) \vee \text{LK}(T))$$

Where

Property	Definition
LD	$\forall a, b, c : c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$
LC	$\forall a, b, c : c \otimes a = c \otimes b \implies a = b$
LK	$\forall a, b, c : c \otimes a = c \otimes b$

Prove

$\text{LD}(S) \wedge \text{LD}(T) \wedge (\text{LC}(S) \vee \text{LK}(T)) \implies \text{LD}(S \vec{\times} T)$

Assume S and T are bisemigroups, $\text{LD}(S) \wedge \text{LD}(T) \wedge (\text{LC}(S) \vee \text{LK}(T))$, and

$$(s_1, t_1), (s_2, t_2), (s_3, t_3) \in S \times T.$$

Then (dropping operator subscripts for clarity) we have

$$\begin{array}{lcl} \text{lhs} & = & (s_1, t_1) \otimes ((s_2, t_2) \vec{\oplus} (s_3, t_3)) \\ & = & (s_1, t_1) \otimes (s_2 \oplus s_3, t_{\text{lhs}}) \\ & = & (s_1 \otimes (s_2 \oplus s_3), t_1 \otimes t_{\text{lhs}}) \end{array}$$

$$\begin{array}{lcl} \text{rhs} & = & ((s_1, t_1) \otimes (s_2, t_2)) \vec{\oplus} ((s_1, t_1) \otimes (s_3, t_3)) \\ & = & (s_1 \otimes s_2, t_1 \otimes t_2) \vec{\oplus} (s_1 \otimes s_3, t_1 \otimes t_3) \\ & = & ((s_1 \otimes s_2) \oplus_S (s_1 \otimes s_3), t_{\text{rhs}}) \\ & = & (s_1 \otimes (s_2 \oplus s_3), t_{\text{rhs}}) \end{array}$$

where t_{lhs} and t_{rhs} are determined by the definition of $\vec{\oplus}$.

We need to show that $lhs = rhs$, that is $t_{\text{rhs}} = t_1 \otimes t_{\text{lhs}}$.

Case 1 : $\text{LC}(S)$

Note that we have

$$(\star) \quad \forall a, b, c : a \neq b \implies c \otimes a \neq c \otimes b$$

Case 1.1 : $s_2 = s_2 \oplus s_3 = s_3$. Then $t_{\text{lhs}} = t_2 \oplus t_3$ and $t_1 \otimes t_{\text{lhs}} = t_1 \otimes (t_2 \oplus t_3) = (t_1 \otimes t_2) \oplus (t_1 \otimes t_3)$, by $\text{LD}(S)$. Also, $s_1 \otimes_S s_2 = s_1 \otimes_S s_3$ and $s_1 \otimes s_2 = s_1 \otimes (s_2 \oplus s_3) = (s_1 \otimes s_2) \oplus (s_1 \otimes s_3)$, again by $\text{LD}(S)$. Therefore $t_{\text{rhs}} = (t_1 \otimes t_2) \oplus (t_1 \otimes t_3) = t_1 \otimes t_{\text{lhs}}$.

Case 1.2 : $s_2 = s_2 \oplus s_3 \neq s_3$. Then $t_1 \otimes t_{\text{lhs}} = t_1 \otimes t_2$ Also
$s_2 = s_2 \oplus s_3 \implies s_1 \otimes s_2 = s_1 \otimes (s_2 \oplus s_3)$ and by $\star$
$s_2 \oplus s_3 \neq s_3 \implies s_1 \otimes (s_2 \oplus s_3) \neq s_1 \otimes s_3$. Thus, by $\text{LD}(S)$,
$(s_1 \otimes s_2) \oplus (s_1 \otimes s_3) \neq s_1 \otimes s_3$ and we get $t_{\text{rhs}} = t_1 \otimes t_2 = t_1 \otimes t_{\text{lhs}}$.

Case 1.3 : $s_2 \neq s_2 \oplus_S s_3 = s_3$. Similar to case 1.2.

Case 2 : $\text{LK}(T)$

Case 2.1 : $s_2 = s_2 \oplus_S s_3 = s_3$. Same as Case 1.1.

Case 2.2 : $s_2 = s_2 \oplus_S s_3 \neq s_3$. Then $t_1 \otimes t_{\text{lhs}} = t_1 \otimes t_2$. Now, $(s_1 \otimes s_2) \oplus_S (s_1 \otimes s_3) = s_1 \otimes (s_2 \oplus s_3) = s_1 \otimes s_2$. So $t_{\text{rhs}} = (t_1 \otimes t_2) \oplus (t_1 \otimes t_3) = t_1 \otimes (t_2 \oplus t_3)$ or $t_{\text{rhs}} = (t_1 \otimes t_2)$. In either case, t_{rhs} is of the form $t_1 \otimes t$, so by $\text{LK}(T)$ we know that $t_{\text{rhs}} = t_1 \otimes t_{\text{lhs}}$.

Case 2.3 : $s_2 \neq s_2 \oplus_S s_3 = s_3$. Similar to case 2.2.

Examples

name	LD	LC	LK
min_plus	Yes	Yes	No
max_min	Yes	No	No
sep(G)	Yes	No	No

So we have

$$\mathsf{LD}(\text{min\_plus} \vec{\times} \text{max\_min})$$
$$\mathsf{LD}(\text{min\_plus} \vec{\times} \text{sep}(G))$$

But

$$\neg(\mathsf{LD}(\text{max\_min} \vec{\times} \text{min\_plus}))$$
$$\neg(\mathsf{LD}(\text{sep}(G) \vec{\times} \text{min\_plus}))$$

Shortest paths with best paths

Let's use

$$\text{add\_zero}(\infty,\ \text{min\_plus} \vec{\times} \text{sep}(G))$$

$$\mathbf{I} \quad = \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} (0, \{\epsilon\}) & \infty & \infty & \infty & \infty \\ \infty & (0, \{\epsilon\}) & \infty & \infty & \infty \\ \infty & \infty & (0, \{\epsilon\}) & \infty & \infty \\ \infty & \infty & \infty & (0, \{\epsilon\}) & \infty \\ \infty & \infty & \infty & \infty & (0, \{\epsilon\}) \end{array} \right] \end{array}$$

$$\mathbf{A} \quad = \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & \end{array} \\ \left[\begin{array}{ccccc} \infty & (2, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (6, \{[(0,3)]\}) & \\ (2, \{[(1,0)]\}) & \infty & (5, \{[(1,2)]\}) & \infty & (4, \{[\\ (1, \{[(2,0)]\}) & (5, \{[(2,1)]\}) & \infty & (4, \{[(2,3)]\}) & (3, \{[\\ (6, \{[(3,0)]\}) & \infty & (4, \{[(3,2)]\}) & \infty & \\ \infty & (4, \{[(4,1)]\}) & (3, \{[(4,2)]\}) & \infty & \end{array} \right. \end{array}$$

Solution

$$
\mathbf{A}^* = \begin{array}{c c}
 & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} &
\left[\begin{array}{ccc}
(0, \{\epsilon\}) & (2, \{[(0,1)]\}) & (1, \{[(0,2)]\}) \\
(2, \{[(1,0)]\}) & (0, \{\epsilon\}) & (3, \{[(1,0),(0,2)]\}) \\
(1, \{[(2,0)]\}) & (3, \{[(2,0),(0,1)]\}) & (0, \{\epsilon\}) \\
(5, \{[(3,2),(2,0)]\}) & (7, \{[(3,2),(2,0),(0,1)]\}) & (4, \{[(3,2)]\}) \\
(4, \{[(4,2),(2,0)]\}) & (4, \{[(4,1)]\}) & (3, \{[(4,2)]\})
\end{array}\right.
\end{array}
$$

Starting in an arbitrary state? No!

Let's use our friend

$$\text{add\_zero}(\infty,\ \text{min\_plus} \vec{\times} \text{sep}(G))$$

Problem:

$$\begin{array}{c} \vdots \quad \vdots \quad \vdots \\ \mathbf{B}_{998} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} (0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (999, \{\}) \\ (1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & (999, \{\}) \\ (1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (999, \{\}) \\ \infty & \infty & \infty & (0, \{\epsilon\}) \end{bmatrix} \end{array} \\ \vdots \quad \vdots \quad \vdots \end{array}$$

Starting in an arbitrary state?

Solution: use another reduction!

$$\begin{array}{rcl} r(\infty) & = & \infty \\ r(s, W) & = & \begin{cases} \infty & \text{if } W = \{\} \\ (s, W) & \text{otherwise} \end{cases} \end{array}$$

Now use this instead

$$\mathrm{red}_r(\mathrm{add\_zero}(\infty,\ \mathrm{min\_plus} \vec{\times} \mathrm{sep}(G)))$$

Starting in an arbitrary state?

$\mathbf{B}_0$ and $\mathbf{B}_1$

$$
\begin{array}{cc}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} &
\left[\begin{array}{cccc}
(0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (2, \{[(0,1),(1,3 \\
(1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & (1, \{[(1,3)]\} \\
(1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (2, \{[(2,1),(1,3 \\
(2, \{[(3,1),(1,0)]\}) & (1, \{[(3,1)]\}) & (2, \{[(3,1),(1,2)]\}) & (0, \{\epsilon\})
\end{array}\right.
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} &
\left[\begin{array}{cccc}
(0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (2, \{[(0,1),(1,3)]\}) \\
(1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & \infty \\
(1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (2, \{[(2,1),(1,3)]\}) \\
\infty & \infty & \infty & (0, \{\epsilon\})
\end{array}\right]
\end{array}
$$

Starting in an arbitrary state?

$\mathbf{B}_2$ and $\mathbf{B}_3$

$$
\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} &
\begin{bmatrix}
(0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (3, \{[(0,2),(2,1),(1,3)]\}) \\
(1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & \infty \\
(1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (3, \{[(2,0),(0,1),(1,3)]\}) \\
\infty & \infty & \infty & (0, \{\epsilon\})
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} &
\begin{bmatrix}
(0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & \infty \\
(1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & \infty \\
(1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & \infty \\
\infty & \infty & \infty & (0, \{\epsilon\})
\end{bmatrix}
\end{array}
$$