L11: Algebraic Path Problems with applications to Internet Routing
Lecture 01

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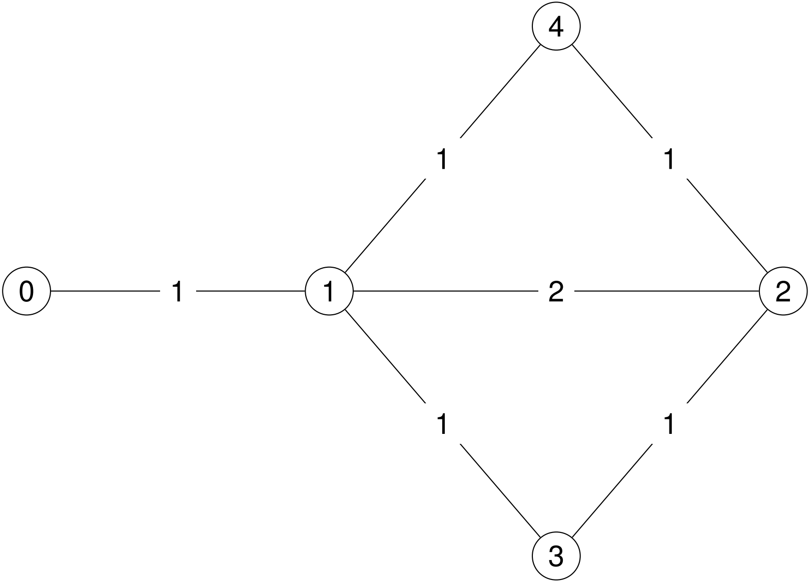
Let's start with shortest paths!



Can represent a problem instance with an adjacency matrix

$$
\mathbf{A} \quad = \quad \begin{array}{c c}
 & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \left[\begin{array}{ccccc}
\infty & 1 & \infty & \infty & \infty \\
1 & \infty & 2 & 1 & 1 \\
\infty & 2 & \infty & 1 & 1 \\
\infty & 1 & 1 & \infty & \infty \\
\infty & 1 & 1 & \infty & \infty
\end{array}\right]
\end{array}
$$

But what problem are we solving?

Classic: globally optimal path weights

We want to find $\mathbf{A}^*$ such that

$$\mathbf{A}^*(i,\ j) = \min_{p \in P(i,\ j)} w(p),$$

where $P(i,\ j)$ is the set of all paths from i to j.

In the example:

$$\mathbf{A}^* \quad = \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 3 & 2 & 2 \\ 1 & 0 & 2 & 1 & 1 \\ 3 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 \\ 2 & 1 & 1 & 2 & 0 \end{array}\right] \end{array}$$

An Algorithm: Dijkstra's

Input : adjacency matrix $\mathbf{A}$ and source vertex $i \in V$,
Output : the i-th row of $\mathbf{R}$, where $\mathbf{R}(i, j)$ is the shortest distance from i to j in the graph represented by $\mathbf{A}$.

(1) **for each** $q \in V$ **do** $\mathbf{R}(i, q) \leftarrow \infty$
(2) $S \leftarrow \{\}$; $\mathbf{R}(i, i) \leftarrow 0$
(3) **while** $S \neq V$ **do**
(4) find $q \in V - S$ such that $\mathbf{R}(i, q)$ is minimal
(5) $S \leftarrow S \cup \{q\}$
(6) **for each** $j \in V - S$ **do**
(7) $\mathbf{R}(i, j) \leftarrow \mathbf{R}(i, j) \min (\mathbf{R}(i, q) + \mathbf{A}(q, j))$

Run this $|V|$ times to get $\mathbf{R} = \mathbf{A}^*$.

But wait! What about the PATHS???

A bit of notation

Assume X and Y are sets of paths over E.

$$X \diamond Y \quad \equiv \quad \{pq \mid p \in X,\ q \in Y\}$$

Dijkstra's Algorithm Augmented With Paths

Input : adjacency matrix $\mathbf{A}$ and source vertex $i \in V$,
Output : the i-th row of $\mathbf{R}$ as before. Now with $\mathbf{P}(i,\ j)$ the set of **all** paths from i to j of distance $\mathbf{R}(i, j)$

$$
\begin{array}{l}
\textbf{for each } q \in V \textbf{ do } \mathbf{R}(i,\ q) \leftarrow \infty;\ \mathbf{P}(i,\ q) \leftarrow \{\} \\
S \leftarrow \{\};\ \mathbf{R}(i,\ i) \leftarrow 0;\ \mathbf{P}(i,\ i) \leftarrow \{\epsilon\} \\
\textbf{while } S \neq V \textbf{ do} \\
\qquad \text{find } q \in V - S \text{ such that } \mathbf{R}(i,\ q) \text{ is minimal} \\
\qquad S \leftarrow S \cup \{q\} \\
\qquad \textbf{for each } j \in V - S \textbf{ do} \\
\qquad\quad \textbf{if } \mathbf{R}(i,\ j) = \mathbf{R}(i,\ q) + \mathbf{A}(q,\ j) \\
\qquad\quad \textbf{then } \mathbf{P}(i,\ j) \leftarrow \mathbf{P}(i,\ j) \cup (\mathbf{P}(i,\ q) \diamond \{(q,\ j)\}) \\
\qquad\quad \textbf{else if } \mathbf{R}(i,\ j) > \mathbf{R}(i,\ q) + \mathbf{A}(q,\ j) \\
\qquad\quad \textbf{then } \mathbf{R}(i,\ j) \leftarrow \mathbf{R}(i,\ q) + \mathbf{A}(q,\ j); \\
\qquad\qquad\qquad \mathbf{P}(i,\ j) \leftarrow \mathbf{P}(i,\ q) \diamond \{(q,\ j)\}
\end{array}
$$

Solution(s)

$$
\mathbf{R} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 3 & 2 & 2 \\ 1 & 0 & 2 & 1 & 1 \\ 3 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 2 \\ 2 & 1 & 1 & 2 & 0 \end{bmatrix} \end{array}
$$

$$
\begin{array}{lcl}
\mathbf{P}(0,0) & = & \{\epsilon\} \\
\mathbf{P}(0,1) & = & \{(0,1)\} \\
\mathbf{P}(0,2) & = & \{(0,1,2),\ (0,1,3,2),\ (0,1,4,2)\} \\
\mathbf{P}(2,1) & = & \{(2,1),\ (2,3,1),\ (2,4,1)\} \\
\mathbf{P}(2,0) & = & \{(2,1,0),\ (2,3,1,0),\ (2,4,1,0)\} \\
\vdots & \vdots & \vdots
\end{array}
$$

Note : could use just the next hop to implement hop-by-hop forwarding.

Let's enrich the metric to *Widest Shortest-Paths*

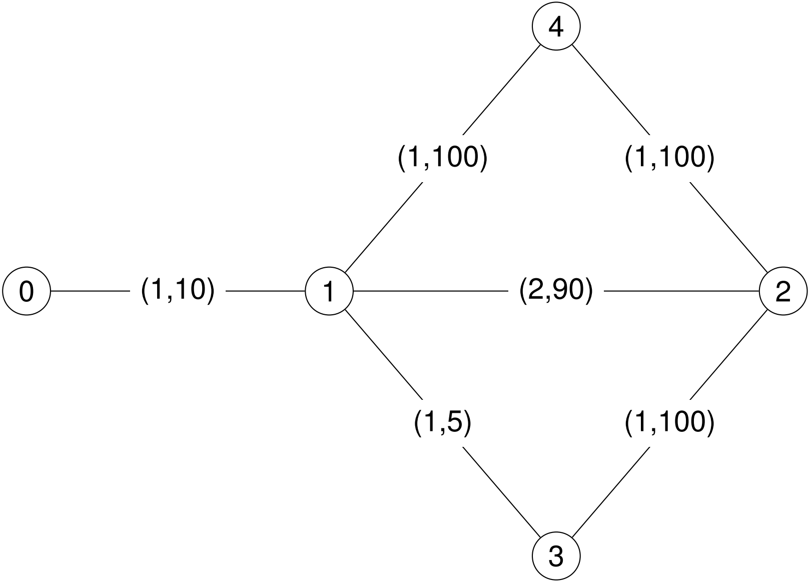
shortest paths	widest shortest paths
$\mathbb{N} \cup \{\infty\}$	$S_{\mathrm{wsp}} \equiv (\mathbb{N} \times \{1, \ldots, \top\}) \cup \{\infty\}$
min	$\circ$
$+$	$\bullet$
0	$(0, \top)$

Can replace $+$ by $\bullet$ and min by $\circ$ in both Dijkstra and Bellman-Ford.

$$(a, b) \circ (c, d) = \begin{cases} (a, b \max d) & (a = c) \\ (a, b) & (a < c) \\ (c, d) & (c < a) \end{cases}$$

$$(a, b) \bullet (c, d) = (a + c, b \min d)$$

Add bandwith to link weights



Weights are globally optimal

Widest shortest-path weights computed by Dijkstra and Bellman-Ford

$$\mathbf{R} \quad = \quad \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & (0, \top) & (1, 10) & (3, 10) & (2, 5) & (2, 10) \\ 1 & (1, 10) & (0, \top) & (2, 100) & (1, 5) & (1, 100) \\ 2 & (3, 10) & (2, 100) & (0, \top) & (1, 100) & (1, 100) \\ 3 & (2, 5) & (1, 5) & (1, 100) & (0, \top) & (2, 100) \\ 4 & (2, 10) & (1, 100) & (1, 100) & (2, 100) & (0, \top) \end{array}$$

Four optimal paths of weight $(3, 10)$. Do our algorithms find all of them?

$$\begin{array}{rcl} \mathbf{P}_{\text{optimal}}(0, 2) & = & \{(0, 1, 2),\ (0, 1, 4, 2)\} \\ \mathbf{P}_{\text{optimal}}(2, 0) & = & \{(2, 1, 0),\ (2, 4, 1, 0)\} \end{array}$$

Surprise!

Four **optimal** paths of weight (3, 10)

$$\begin{array}{rcl} \mathbf{P}_{\text{optimal}}(0,2) & = & \{(0,1,2),\ (0,1,4,2)\} \\ \mathbf{P}_{\text{optimal}}(2,0) & = & \{(2,1,0),\ (2,4,1,0)\} \end{array}$$

Paths computed by **Dijkstra**

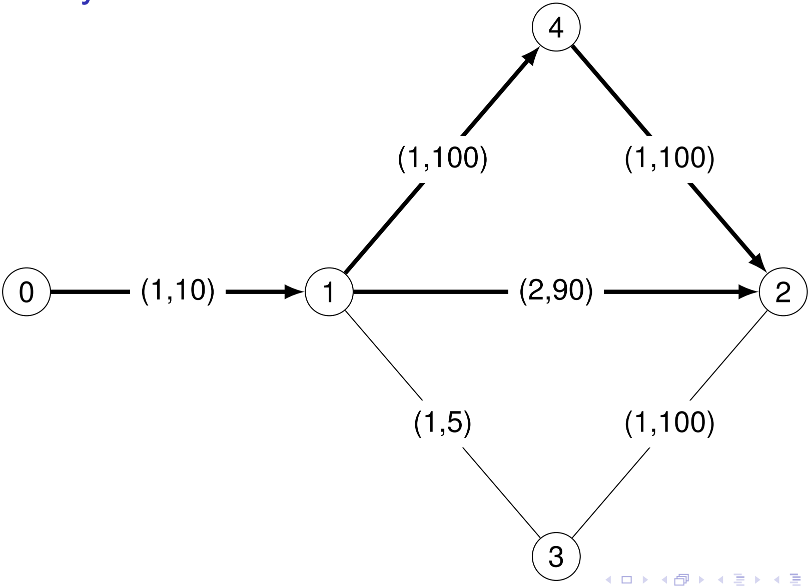
$$\begin{array}{rcl} \mathbf{P}_{\text{Dijkstra}}(0,2) & = & \{(0,1,2),\ (0,1,4,2)\} \\ \mathbf{P}_{\text{Dijkstra}}(2,0) & = & \{(2,4,1,0)\} \end{array}$$

Notice that 0's paths cannot both be implemented with next-hop forwarding since $\mathbf{P}_{\text{Dijkstra}}(1,2) = \{(1,4,2)\}$.

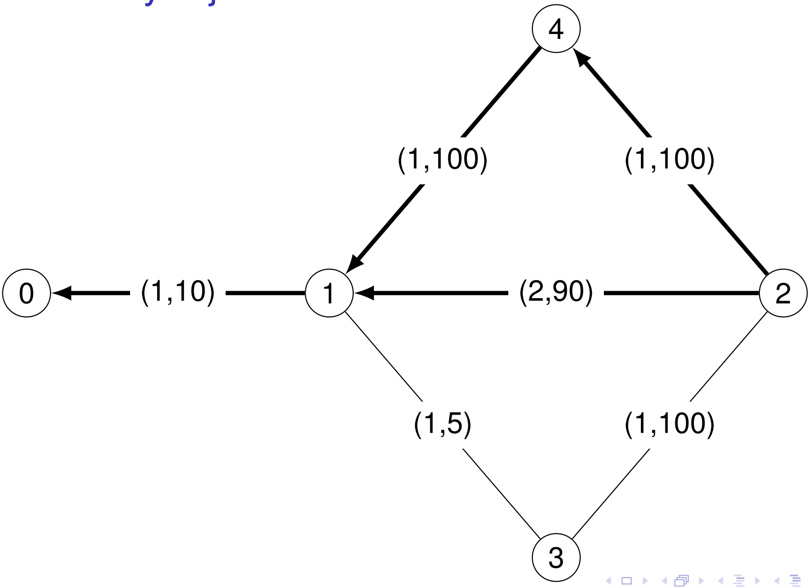
Paths computed by **Distributed Bellman-Ford** (Explained in later lectures)

$$\begin{array}{rcl} \mathbf{P}_{\text{Bellman}}(0,2) & = & \{(0,1,4,2)\} \\ \mathbf{P}_{\text{Bellman}}(2,0) & = & \{(2,1,0),\ (2,4,1,0)\} \end{array}$$

Optimal paths from 0 to 2. Computed by Dijkstra but not by Bellman-Ford



Optimal paths from 2 to 1. Computed by Bellman-Ford but not by Dijkstra



Observations

For distributed Bellman-Ford

$$\begin{array}{rcl} \textit{next-hop-paths}(\mathbf{A}) & = & \textit{computed-paths}(\mathbf{A}) \\ & \subseteq & \textit{optimal-paths}(\mathbf{A}) \end{array}$$

For Dijkstra's algorithm

$$\begin{array}{rcl} \textit{next-hop-paths}(\mathbf{A}) & \subseteq & \textit{computed-paths}(\mathbf{A}) \\ & \subseteq & \textit{optimal-paths}(\mathbf{A}) \end{array}$$

We will see that all of these path sets coincide exactly when the metric is *cancellative*. That is, when $a \otimes b = a \otimes c$ always implies that $b = c$.

What is going on here???

Help!

- Are the algorithms broken?
- Is the new metric broken?

L11

This course will provide you with the tools to answert these questions!

see also ....

On the Forwarding Paths Produced by Internet Routing Algorithms. Seweryn Dynerowicz and Timothy G. Griffin. To be presented at ICNP 2013 on 10 October, 2013.

Our approach

The Algorithm to Algebra (A2A) method

$$\begin{pmatrix} \text{original metric} \\ + \\ \text{complex algorithm} \end{pmatrix} \rightarrow \begin{pmatrix} \text{modified metric} \\ + \\ \text{generic algorithm} \end{pmatrix}$$

A2A attempts to shift complexity from an algorithm to the metric, which is captured in an algebraic structure such as a semiring.

The **Tentative** Plan

1	11 October	:	The Paths Puzzle
2	16 October	:	Semigroups and Order Relations
3	18 October	:	Semirings — Theory
4	23 October	:	Semirings — Constructions
5	25 October	:	Semirings — Examples
6	30 October	:	Semirings — algorithms
7	1 November	:	Beyond Semirings — “functions on arcs”
8	6 November	:	Beyond Semirings — Global vs Local optimality
9	8 November	:	Solving the Paths Puzzle (**HW 1 due**)
10	13 November	:	Graph (Network) decomposition
11	15 November	:	Protocols : RIP, OSPF, IS-IS
12	20 November	:	More on Global vs Local optimality
13	23 November	:	Protocols : EIGRP, BGP
14	27 November	:	Dijkstra revisited
15	29 November	:	Route redistribution, administrative distance
16	4 December	:	Metarouting project (**HW 2 due**)
	15 January	:	**HW 3 due**

Semigroups

Definition (Semigroup)

A semigroup $(S, \oplus)$ is a non-empty set S with a binary operation such that

$$\text{ASSOCIATIVE} \quad : \quad a \oplus (b \oplus c) = (a \oplus b) \oplus c$$

S	$\oplus$	where
$\mathbb{N}^{\infty}$	min	
$\mathbb{N}^{\infty}$	max	
$\mathbb{N}^{\infty}$	$+$	
2^W	$\cup$	
2^W	$\cap$	
S^*	$\circ$	$(abc \circ de = abcde)$
S	left	$(a \text{ left } b = a)$
S	right	$(a \text{ right } b = b)$

Special Elements

Definition

- $\alpha \in S$ is an identity if for all $a \in S$

$$a = \alpha \oplus a = a \oplus \alpha$$

- A semigroup is a monoid if it has an identity.
- ω is an annihilator if for all $a \in S$

$$\omega = \omega \oplus a = a \oplus \omega$$

S	$\oplus$	α	ω
$\mathbb{N}^\infty$	min	∞	0
$\mathbb{N}^\infty$	max	0	∞
$\mathbb{N}^\infty$	$+$	0	∞
2^W	$\cup$	$\{\}$	W
2^W	$\cap$	W	$\{\}$
S^*	$\circ$	ϵ	
S	left		
S	right		

Important Properties

Definition (Some Important Semigroup Properties)

$$\begin{array}{rcrcl} \text{COMMUTATIVE} & : & a \oplus b & = & b \oplus a \\ \text{SELECTIVE} & : & a \oplus b & \in & \{a,\ b\} \\ \text{IDEMPOTENT} & : & a \oplus a & = & a \end{array}$$

S	$\oplus$	COMMUTATIVE	SELECTIVE	IDEMPOTENT
$\mathbb{N}^{\infty}$	min	$\star$	$\star$	$\star$
$\mathbb{N}^{\infty}$	max	$\star$	$\star$	$\star$
$\mathbb{N}^{\infty}$	$+$	$\star$		
2^W	$\cup$	$\star$		$\star$
2^W	$\cap$	$\star$		$\star$
S^*	$\circ$			
S	left		$\star$	$\star$
S	right		$\star$	$\star$

Order Relations

We are interested in order relations $\leq \subseteq S \times S$

Definition (Important Order Properties)

$$\begin{array}{rcl} \text{REFLEXIVE} & : & a \leq a \\ \text{TRANSITIVE} & : & a \leq b \wedge b \leq c \rightarrow a \leq c \\ \text{ANTISYMMETRIC} & : & a \leq b \wedge b \leq a \rightarrow a = b \\ \text{TOTAL} & : & a \leq b \vee b \leq a \end{array}$$

	pre-order	partial order	preference order	total order
REFLEXIVE	$\star$	$\star$	$\star$	$\star$
TRANSITIVE	$\star$	$\star$	$\star$	$\star$
ANTISYMMETRIC		$\star$		$\star$
TOTAL			$\star$	$\star$

Canonical Pre-order of a Commutative Semigroup

Suppose $\oplus$ is commutative.

Definition (Canonical pre-orders)

$$
\begin{aligned}
a \trianglelefteq_{\oplus}^{R} b &\equiv \exists c \in S : b = a \oplus c \\
a \trianglelefteq_{\oplus}^{L} b &\equiv \exists c \in S : a = b \oplus c
\end{aligned}
$$

Lemma (Sanity check)

Associativity of $\oplus$ implies that these relations are transitive.

Proof.

Note that $a \trianglelefteq_{\oplus}^{R} b$ means $\exists c_1 \in S : b = a \oplus c_1$, and $b \trianglelefteq_{\oplus}^{R} c$ means $\exists c_2 \in S : c = b \oplus c_2$. Letting $c_3 = c_1 \oplus c_2$ we have $c = b \oplus c_2 = (a \oplus c_1) \oplus c_2 = a \oplus (c_1 \oplus c_2) = a \oplus c_3$. That is, $\exists c_3 \in S : c = a \oplus c_3$, so $a \trianglelefteq_{\oplus}^{R} c$. The proof for $\trianglelefteq_{\oplus}^{L}$ is similar. $\square$

Canonically Ordered Semigroup

Definition (Canonically Ordered Semigroup)

A commutative semigroup $(S, \oplus)$ is canonically ordered when $a \trianglelefteq^R_\oplus c$ and $a \trianglelefteq^L_\oplus c$ are partial orders.

Definition (Groups)

A monoid is a group if for every $a \in S$ there exists a $a^{-1} \in S$ such that $a \oplus a^{-1} = a^{-1} \oplus a = \alpha$.

Canonically Ordered Semigroups vs. Groups

Lemma (THE BIG DIVIDE)

Only a trivial group is canonically ordered.

Proof.

If $a, \ b \in S$, then $a = \alpha_{\oplus} \oplus a = (b \oplus b^{-1}) \oplus a = b \oplus (b^{-1} \oplus a) = b \oplus c$, for $c = b^{-1} \oplus a$, so $a \trianglelefteq^{L}_{\oplus} b$. In a similar way, $b \trianglelefteq^{R}_{\oplus} a$. Therefore $a = b$. $\square$

Natural Orders

Definition (Natural orders)

Let $(S, \oplus)$ be a semigroup.

$$
\begin{array}{rcl}
a \leq^{L}_{\oplus} b & \equiv & a = a \oplus b \\
a \leq^{R}_{\oplus} b & \equiv & b = a \oplus b
\end{array}
$$

Lemma

If $\oplus$ is commutative and idempotent, then $a \trianglelefteq^{D}_{\oplus} b \iff a \leq^{D}_{\oplus} b$, for $D \in \{R, L\}$.

Proof.

$$
\begin{array}{rcl}
a \trianglelefteq^{R}_{\oplus} b & \iff & b = a \oplus c = (a \oplus a) \oplus c = a \oplus (a \oplus c) \\
& = & a \oplus b \iff a \leq^{R}_{\oplus} b \\
a \trianglelefteq^{L}_{\oplus} b & \iff & a = b \oplus c = (b \oplus b) \oplus c = b \oplus (b \oplus c) \\
& = & b \oplus a = a \oplus b \iff a \leq^{L}_{\oplus} b
\end{array}
$$

$\square$

Special elements and natural orders

Lemma (Natural Bounds)

- *If α exists, then for all a, $a \leq^L_\oplus \alpha$ and $\alpha \leq^R_\oplus a$*
- *If ω exists, then for all a, $\omega \leq^L_\oplus a$ and $a \leq^R_\oplus \omega$*
- *If α and ω exist, then S is bounded.*

$$\begin{array}{ccccc} \omega & \leq^L_\oplus & a & \leq^L_\oplus & \alpha \\ \alpha & \leq^R_\oplus & a & \leq^R_\oplus & \omega \end{array}$$

Remark (Thanks to Iljitsch van Beijnum)

Note that this means for (min, $+$) we have

$$\begin{array}{ccccc} 0 & \leq^L_{\text{min}} & a & \leq^L_{\text{min}} & \infty \\ \infty & \leq^R_{\text{min}} & a & \leq^R_{\text{min}} & 0 \end{array}$$

and still say that this is bounded, even though one might argue with the terminology!

Examples of special elements

S	$\oplus$	α	ω	$\leq^{\mathrm{L}}_{\oplus}$	$\leq^{\mathrm{R}}_{\oplus}$
$\mathbb{N} \cup \{\infty\}$	min	∞	0	$\leq$	$\geq$
$\mathbb{N} \cup \{\infty\}$	max	0	∞	$\geq$	$\leq$
$\mathcal{P}(W)$	$\cup$	$\{\}$	W	$\subseteq$	$\supseteq$
$\mathcal{P}(W)$	$\cap$	W	$\{\}$	$\supseteq$	$\subseteq$

Property Management

Lemma

Let $D \in \{R,\ L\}$.

1. $\text{IDEMPOTENT}((S,\ \oplus)) \iff \text{REFLEXIVE}((S,\ \leq^D_\oplus))$
2. $\text{COMMUTATIVE}((S,\ \oplus)) \implies \text{ANTISYMMETRIC}((S,\ \leq^D_\oplus))$
3. $\text{COMMUTATIVE}((S,\ \oplus)) \implies (\text{SELECTIVE}((S,\ \oplus)) \iff \text{TOTAL}((S,\ \leq^D_\oplus)))$

Proof.

1. $a \leq^D_\oplus a \iff a = a \oplus a,$
2. $a \leq^L_\oplus b \wedge b \leq^L_\oplus a \iff a = a \oplus b \wedge b = b \oplus a \implies a = b$
3. $a = a \oplus b \vee b = a \oplus b \iff a \leq^L_\oplus b \vee b \leq^L_\oplus a$

$\square$

Bounds

Suppose $(S, \leq)$ is a partially ordered set.

greatest lower bound

For $a, b \in S$, the element $c \in S$ is the *greatest lower bound of a and b*, written $c = a \,\mathrm{glb}\, b$, if it is a lower bound ($c \leq a$ and $c \leq b$), and for every $d \in S$ with $d \leq a$ and $d \leq b$, we have $d \leq c$.

least upper bound

For $a, b \in S$, the element $c \in S$ is the *least upper bound of a and b*, written $c = a \,\mathrm{lub}\, b$, if it is an upper bound ($a \leq c$ and $b \leq c$), and for every $d \in S$ with $a \leq d$ and $b \leq d$, we have $c \leq d$.

Semi-lattices

Suppose $(S, \leq)$ is a partially ordered set.

meet-semilattice

S is a *meet-semilattice* if $a \text{ glb } b$ exists for each $a, b \in S$.

join-semilattice

S is a *join-semilattice* if $a \text{ lub } b$ exists for each $a, b \in S$.

Fun Facts

Fact 1

Suppose $(S, \oplus)$ is a commutative and idempotent semigroup.

- $(S, \leq^L_\oplus)$ is a meet-semilattice with $a \,\mathrm{glb}\, b = a \oplus b$.
- $(S, \leq^R_\oplus)$ is a join-semilattice with $a \,\mathrm{lub}\, b = a \oplus b$.

Fact 2

Suppose $(S, \leq)$ is a partially ordered set.

- If $(S, \leq)$ is a meet-semilattice, then (S, glb) is a commutative and idempotent semigroup.
- If $(S, \leq)$ is a join-semilattice, then (S, lub) is a commutative and idempotent semigroup.

That is, semi-lattices represent the same class of structures as commutative and idempotent semigroups.

Bi-semigroups and Pre-Semirings

$(S, \oplus, \otimes)$ is a bi-semigroup when

- $(S, \oplus)$ is a semigroup
- $(S, \otimes)$ is a semigroup

$(S, \oplus, \otimes)$ is a pre-semiring when

- $(S, \oplus, \otimes)$ is a bi-semigroup
- $\oplus$ is commutative

and left- and right-distributivity hold,

$$
\begin{array}{lcrcl}
\mathrm{LD} & : & a \otimes (b \oplus c) & = & (a \otimes b) \oplus (a \otimes c) \\
\mathrm{RD} & : & (a \oplus b) \otimes c & = & (a \otimes c) \oplus (b \otimes c)
\end{array}
$$

Semirings

$(S, \oplus, \otimes, \overline{0}, \overline{1})$ is a semiring when

- $(S, \oplus, \otimes)$ is a pre-semiring
- $(S, \oplus, \overline{0})$ is a (commutative) monoid
- $(S, \otimes, \overline{1})$ is a monoid
- $\overline{0}$ is an annihilator for $\otimes$

Examples

Pre-semirings

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
min_plus	$\mathbb{N}$	min	$+$		0
max_min	$\mathbb{N}$	max	min	0	

Semirings

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
sp	$\mathbb{N}^{\infty}$	min	$+$	∞	0
bw	$\mathbb{N}^{\infty}$	max	min	0	∞

Note the sloppiness — the symbols $+$, max, and min in the two tables represent different functions....

How about $(\max, +)$?

Pre-semiring

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
$\mathrm{max\_plus}$	$\mathbb{N}$	max	$+$	0	0

- What about "$\overline{0}$ is an annihilator for $\otimes$"? No!

Semiring ($\mathrm{max\_plus}^{-\infty} = \mathrm{add\_zero}(-\infty,\ \mathrm{max\_min})$)

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
$\mathrm{max\_plus}^{-\infty}$	$\mathbb{N} \cup \{-\infty\}$	max	$+$	$-\infty$	0

Matrix Semirings

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ a semiring
- Define the semiring of $n \times n$-matrices over S : $(\mathbb{M}_n(S), \oplus, \otimes, \mathbf{J}, \mathbf{I})$

$\oplus$ and $\otimes$

$$\begin{aligned}(\mathbf{A} \oplus \mathbf{B})(i, j) &= \mathbf{A}(i, j) \oplus \mathbf{B}(i, j) \\ (\mathbf{A} \otimes \mathbf{B})(i, j) &= \bigoplus_{1 \le q \le n} \mathbf{A}(i, q) \otimes \mathbf{B}(q, j)\end{aligned}$$

J and I

$$\begin{aligned}\mathbf{J}(i, j) &= \overline{0} \\ \mathbf{I}(i, j) &= \begin{cases} \overline{1} & (\text{if } i = j) \\ \overline{0} & (\text{otherwise}) \end{cases}\end{aligned}$$

$\mathbb{M}_n(S)$ is a semiring!

For example, here is left distribution

$$\mathbf{A} \otimes (\mathbf{B} \oplus \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \oplus (\mathbf{A} \otimes \mathbf{C})$$

$$\begin{aligned}
& (\mathbf{A} \otimes (\mathbf{B} \oplus \mathbf{C}))(i,\ j) \\
= & \bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes (\mathbf{B} \oplus \mathbf{C})(q,\ j) \\
= & \bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes (\mathbf{B}(q,\ j) \oplus \mathbf{C}(q,\ j)) \\
= & \bigoplus_{1 \le q \le n} (\mathbf{A}(i,\ q) \otimes \mathbf{B}(q,\ j)) \oplus (\mathbf{A}(i,\ q) \otimes \mathbf{C}(q,\ j)) \\
= & (\bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes \mathbf{B}(q,\ j)) \oplus (\bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes \mathbf{C}(q,\ j)) \\
= & ((\mathbf{A} \otimes \mathbf{B}) \oplus (\mathbf{A} \otimes \mathbf{C}))(i,\ j)
\end{aligned}$$

Note : we only needed left-distributivity on S.

Matrix encoding path problems

- $(S,\ \oplus,\ \otimes,\ \overline{0},\ \overline{1})$ a semiring
- $G = (V,\ E)$ a directed graph
- $w \in E \to S$ a weight function

Path weight

The *weight* of a path $p = i_1, i_2, i_3, \cdots, i_k$ is

$$w(p) = w(i_1,\ i_2) \otimes w(i_2,\ i_3) \otimes \cdots \otimes w(i_{k-1},\ i_k).$$

The empty path is given the weight $\overline{1}$.

Adjacency matrix **A**

$$\mathbf{A}(i,\ j) = \begin{cases} w(i,\ j) & \text{if } (i,\ j) \in E, \\ \overline{0} & \text{otherwise} \end{cases}$$

The general problem of finding globally optimal paths

Given an adjacency matrix $\mathbf{A}$, find $\mathbf{R}$ such that for all $i, j \in V$

$$\mathbf{R}(i, j) = \bigoplus_{p \in P(i, j)} w(p)$$

How can we solve this problem?

Matrix methods

Matrix powers, $\mathbf{A}^k$

$$\mathbf{A}^0 = \mathbf{I}$$

$$\mathbf{A}^{k+1} = \mathbf{A} \otimes \mathbf{A}^k$$

Closure, $\mathbf{A}^*$

$$\mathbf{A}^{(k)} = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k$$

$$\mathbf{A}^* = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k \oplus \cdots$$

Note: $\mathbf{A}^*$ might not exist. Why?

Matrix methods can compute optimal path weights

- Let $P(i, j)$ be the set of paths from i to j.
- Let $P^k(i, j)$ be the set of paths from i to j with exactly k arcs.
- Let $P^{(k)}(i, j)$ be the set of paths from i to j with at most k arcs.

Theorem

$$
\begin{array}{lrcl}
(1) & \mathbf{A}^k(i,\ j) & = & \bigoplus_{p \in P^k(i,\ j)} w(p) \\
(2) & \mathbf{A}^{(k)}(i,\ j) & = & \bigoplus_{p \in P^{(k)}(i,\ j)} w(p) \\
(3) & \mathbf{A}^*(i,\ j) & = & \bigoplus_{p \in P(i,\ j)} w(p)
\end{array}
$$

Warning again: for some semirings the expression $\mathbf{A}^*(i,\ j)$ might not be well-defeind. Why?

Proof of (1)

By induction on k. Base Case: $k = 0$.

$$P^0(i,\ i) = \{\epsilon\},$$

so $\mathbf{A}^0(i\ , i) = \mathbf{I}(i,\ i) = \overline{1} = w(\epsilon)$.

And $i \neq j$ implies $P^0(i, j) = \{\}$. By convention

$$\bigoplus_{p \in \{\}} w(p) = \overline{0} = \mathbf{I}(i,\ j).$$

Proof of (1)

Induction step.

$$
\begin{aligned}
\mathbf{A}^{k+1}(i,j) &= (\mathbf{A} \otimes \mathbf{A}^{k})(i,\ j) \\
&= \bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes \mathbf{A}^{k}(q,\ j) \\
&= \bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes \Big(\bigoplus_{p \in P^{k}(q,\ j)} w(p)\Big) \\
&= \bigoplus_{1 \le q \le n} \ \bigoplus_{p \in P^{k}(q,\ j)} \mathbf{A}(i,\ q) \otimes w(p) \\
&= \bigoplus_{(i,\ q) \in E} \ \bigoplus_{p \in P^{k}(q,j)} w(i,\ q) \otimes w(p) \\
&= \bigoplus_{p \in P^{k+1}(i,\ j)} w(p)
\end{aligned}
$$

When does $\mathbf{A}^{(*)}$ exist? Try a general approach.

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ a semiring

Powers, a^k

$$\begin{aligned} a^0 &= \overline{1} \\ a^{k+1} &= a \otimes a^k \end{aligned}$$

Closure, a^*

$$\begin{aligned} a^{(k)} &= a^0 \oplus a^1 \oplus a^2 \oplus \cdots \oplus a^k \\ a^* &= a^0 \oplus a^1 \oplus a^2 \oplus \cdots \oplus a^k \oplus \cdots \end{aligned}$$

Definition (q stability)

If there exists a q such that $a^{(q)} = a^{(q+1)}$, then a is q-stable. Therefore, $a^* = a^{(q)}$, assuming $\oplus$ is idempotent.

More Fun Facts

Fact 3

If $\overline{1}$ is an annihiltor for $\oplus$, then every $a \in S$ is 0-stable!

Fact 4

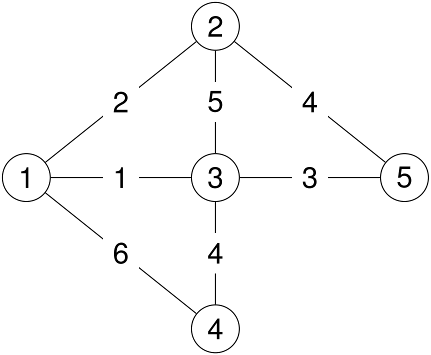
If S is 0-stable, then $\mathbb{M}_n(S)$ is $(n-1)$-stable. That is,

$$\mathbf{A}^* = \mathbf{A}^{(n-1)} = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^{n-1}$$

Homework number 1

- Prove Fun Facts 1, 2, 3, 4.
- Define a *non-commutative* semigroup $(S, \oplus)$ where $\leq^L_\oplus$ is anti-symmetric.

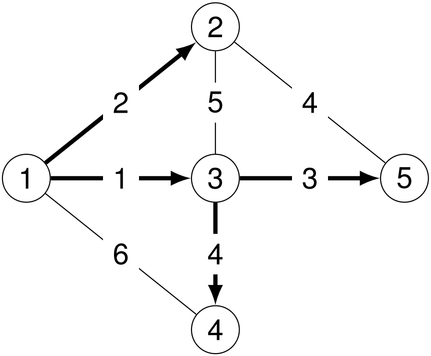
Shortest paths example, $(\mathbb{N}^{\infty},\ \min,\ +)$



The adjacency matrix

$$
\mathbf{A} \quad = \quad
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{ccccc}
\infty & 2 & 1 & 6 & \infty \\
2 & \infty & 5 & \infty & 4 \\
1 & 5 & \infty & 4 & 3 \\
6 & \infty & 4 & \infty & \infty \\
\infty & 4 & 3 & \infty & \infty
\end{array}\right]
\end{array}
$$

Shortest paths example, $(\mathbb{N}^{\infty},\ \min,\ +)$



Bold arrows indicate the shortest-path tree rooted at 1.

The routing matrix

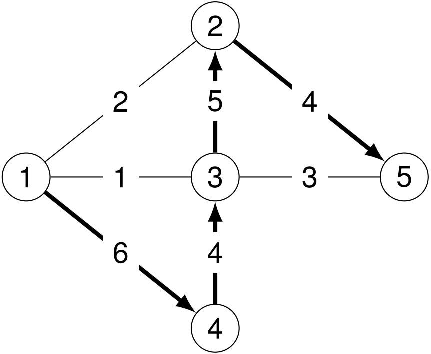
$$\mathbf{R} \quad = \quad \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \left[\begin{array}{ccccc} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{array}\right] \end{array}$$

Matrix **R** solves this global optimality problem:

$$\mathbf{R}(i,\ j) = \min_{p \in P(i,\ j)} w(p),$$

where $P(i,\ j)$ is the set of all paths from i to j.

Widest paths example, $(\mathbb{N}^{\infty}, \max, \min)$



Bold arrows indicate the widest-path tree rooted at 1.

The routing matrix

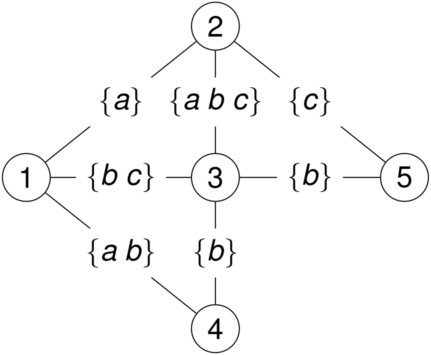
$$\mathbf{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \infty & 4 & 4 & 6 & 4 \\ 4 & \infty & 5 & 4 & 4 \\ 4 & 5 & \infty & 4 & 4 \\ 6 & 4 & 4 & \infty & 4 \\ 4 & 4 & 4 & 4 & \infty \end{bmatrix} \end{array}$$

Matrix **R** solves this global optimality problem:

$$\mathbf{R}(i,\ j) = \max_{p \in P(i,\ j)} w(p),$$

where $w(p)$ is now the minimal edge weight in p.

Unfamiliar example, $(2^{\{a,\ b,\ c\}},\ \cup,\ \cap)$



We want a Matrix **R** to solve this global optimality problem:

$$\mathbf{R}(i,\ j) = \bigcup_{p \in P(i,\ j)} w(p),$$

where $w(p)$ is now the intersection of all edge weights in p.

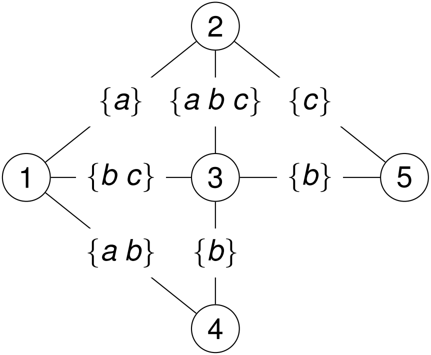
For $x \in \{a,\ b,\ c\}$, interpret $x \in \mathbf{R}(i,\ j)$ to mean that there is at least one path from i to j with x in every arc weight along the path.

Unfamiliar example, $(2^{\{a,\ b,\ c\}},\ \cup,\ \cap)$

The matrix **R**

$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad & \quad 5 \quad \end{array} \\
\left[\begin{array}{ccccc}
\{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\} & \{b\ c\} \\
\{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\} & \{b\ c\} \\
\{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\ c\} & \{a\ b\} & \{b\ c\} \\
\{a\ b\} & \{a\ b\} & \{a\ b\} & \{a\ b\ c\} & \{b\} \\
\{b\ c\} & \{b\ c\} & \{b\ c\} & \{b\} & \{a\ b\ c\}
\end{array}\right]
\end{array}
$$

Another unfamiliar example, $(2^{\{a,\ b,\ c\}},\ \cap,\ \cup)$



We want matrix **R** to solve this global optimality problem:

$$\mathbf{R}(i,\ j) = \bigcap_{p \in P(i,\ j)} w(p),$$

where $w(p)$ is now the union of all edge weights in p.

For $x \in \{a,\ b,\ c\}$, interpret $x \in \mathbf{R}(i,\ j)$ to mean that every path from i to j has at least one arc with weight containing x.

Another unfamiliar example, $(2^{\{a,\ b,\ c\}},\ \cap,\ \cup)$

The matrix **R**

$$
\begin{array}{c c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\left[\begin{array}{ccccc}
\{\} & \{\} & \{b\} & \{b\} & \{\} \\
\{\} & \{\} & \{b\} & \{b\} & \{\} \\
\{b\} & \{b\} & \{\} & \{b\} & \{b\} \\
\{b\} & \{b\} & \{b\} & \{\} & \{b\} \\
\{\} & \{\} & \{b\} & \{b\} & \{\}
\end{array} \right]
\end{array}
$$