L108: Category theory and logic
Exercise sheet 4

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Finite limits

Finite products belong to a larger class of constructions known as *finite limits*. On this exercise sheet, we consider two other kinds of finite limits, called *equalizers* and *pullbacks*.

1. **Equalizers**. Let $\mathbb{C}$ be a category, and let $f, g : A \rightrightarrows B$ be two parallel morphisms in $\mathbb{C}$. An *equalizer of f and g* is a morphism $m : U \to A$ such that $fm = gm$, and for every $h : X \to A$ with $fh = gh$ there exists a unique $k : X \to U$ with $mk = h$.

$$\begin{tikzcd} X \arrow[d, dashed, "k"'] \arrow[dr, "h"] & & \\ U \arrow[r, "m"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

   (a) Show that every equalizer is a monomorphism.

   (b) Given f, g as above, define a category in which equalizers of f and g are terminal objects, and deduce that equalizers are unique up to unique isomorphism.

   (c) In **Set**, all parallel pairs of morphisms have equalizers. Given two functions $f, g : A \rightrightarrows B$, give a construction of the equalizer.

2. **Pullbacks**. Let $\mathbb{C}$ be a category, and let $A \xrightarrow{f} C \xleftarrow{g} B$ be a cospan in $\mathbb{C}$. A *pullback*[1] of $A \xrightarrow{f} C \xleftarrow{g} B$ is a span $A \xleftarrow{p} P \xrightarrow{q} B$ such that

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \tag{1}$$

   commutes, and for every span $A \xleftarrow{h} X \xrightarrow{k} B$ with $fh = gk$ there exists a unique $m : X \to B$ with $pm = h$ and $qm = k$.

$$\begin{tikzcd} X \arrow[dr, dashed, "m"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "k"'] & & \\ & P \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd} \tag{2}$$

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[1] sometimes also called *fibered product*

In this case we also say that (1) is a *pullback square*.

(a) Define a category in which pullbacks of $A \xrightarrow{f} C \xleftarrow{g} B$ are terminal objects, and deduce that pullbacks are unique up to unique isomorphism.

(b) In **Set** all pullbacks exist. Give a construction.

(c) Consider a commutative diagram of the form

$$\begin{tikzcd} Q \arrow[r, "q"] \arrow[d, "p"'] & P \arrow[r, "s"] \arrow[d, "r"'] & C \arrow[d, "h"] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & D \end{tikzcd}$$

in a category $\mathbb{C}$, and assume that the right square is a pullback. Show that the left square is a pullback if and only if the outer rectangle is a pullback. (This is known as the *pullback lemma*.)

Semantics of simply typed lambda calculus

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3. The axioms for product types in the equational theory of the simply typed λ-calculus are the *β-rules*

$$\mathrm{fst}(s, t) = s \qquad \text{and} \qquad \mathrm{snd}(s, t) = t$$

and the *η-rule*

$$u = (\mathrm{fst}(u), \mathrm{snd}(u))$$

for appropriately typed terms s, t, u in context.

Show that the interpretation of simply typed λ-calculus is sound w.r.t. these rules, in the sense that

– $[\![\Gamma \vdash \mathrm{fst}(s, t) : \alpha]\!] = [\![\Gamma \vdash s : \alpha]\!]$
– $[\![\Gamma \vdash \mathrm{snd}(s, t) : \beta]\!] = [\![\Gamma \vdash t : \beta]\!]$
– $[\![\Gamma \vdash u : \alpha \times \beta]\!] = [\![\Gamma \vdash (\mathrm{fst}(u), \mathrm{snd}(u)) : \alpha \times \beta]\!]$

for all terms $\Gamma \vdash s : \alpha$, $\quad \Gamma \vdash t : \beta$, $\quad \Gamma \vdash u : \alpha \times \beta$.

Cartesian closed categories

The relationship between λ-calculus and cartesian closed categories can be viewed from two perspectives: from the point of view of the *syntax*, cartesian closed categories are *models* for the simply typed λ-calculus. Conversely, from the point of view of *categories*, the λ-calculus can serve as an *internal language* for reasoning in cartesian closed categories.

For example, we can show that two objects A, B of a cartesian closed category are isomorphic by defining terms $x{:}A \vdash t : B$[2] and $y{:}B \vdash x : A$ and showing that the substitutions $x{:}A \vdash s[t/y] : A$ and $y{:}B \vdash t[s/x] : B$ are equal to the 'identity terms' $x{:}A \vdash x : A$ and $y{:}B \vdash y : B$ in the equational theory.

[2]We view objects as type constants here and omit the semantic brackets $[\![\cdot]\!]$.

4. Show that for any three objects X, Y, Z in a cartesian closed category $\mathbb{C}$, there are isomorphisms

   - $X \times Y \cong Y \times X$
   - $(X \times Y)^Z \cong X^Z \times Y^Z$
   - $(X^Y)^Z \cong X^{Y \times Z}$
   - $X^1 \cong X$

   You may do this using the technique sketched above the exercise, but fill in the details and describe *why* this approach shows the claimed isomorphisms.

   For the last isomorphism, you need the η-rule of the unit type, which states that $t = *$ for any term $\Gamma \vdash t : 1$.