L108: Category theory and logic
Exercise sheet 3

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1. **Natural deduction**
   Write proofs in natural deduction of the judgments

   (a) $X \Rightarrow Y, Y \Rightarrow Z \vdash X \Rightarrow Z$

   (b) $Y \vdash (X \Rightarrow Y) \Rightarrow Y$

   (c) $X \vdash (X \Rightarrow Y) \Rightarrow Y$

   (d) $((X \Rightarrow Y) \Rightarrow Y) \Rightarrow Y \vdash X \Rightarrow Y$

   Write down the corresponding terms of simply typed lambda calculus.

2. **Finite products**

   (a) Define a concept of 'n-ary' product $A_1 \times \cdots \times A_n$ ($n \in \mathbb{N}$) of objects $A_1, \ldots, A_n \in \mathrm{obj}(\mathbb{C})$ of a category $\mathbb{C}$, generalizing the binary products introduced in the lecture.

   (b) For $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \mathrm{obj}(\mathbb{C})$, define a category $\mathbf{Span}_{\mathbb{C}}(A_1, \ldots, A_n)$ whose terminal objects are the n-ary products of $A_1, \ldots, A_n$.

   (c) Show that if $\mathbb{C}$ has binary products and a terminal object, then all n-ary products exist.

   (d) Given $A_1, \ldots, A_n, B \in \mathrm{obj}(\mathbb{C})$, construct an isomorphism $j : (A_1 \times \cdots \times A_n) \times B \xrightarrow{\cong} A_1 \times \cdots \times A_n \times B$ (the left expression is a binary product of a of an n-ary product and a single object, while the right expression is an $(n+1)$-ary product).

3. Show that the category **Preord** of preorders is cartesian closed, i.e. show that all finite products and exponential objects exist in **Preord**.

4. **Right monoid actions**
   Let $(M, \cdot, e)$ be a monoid. In the following we will write the multiplication in M as juxtaposition mn instead of $m \cdot n$.

   A *right action*[1] of M on a set X (also called a *right M-action*) is a function

$$X \times M \to X, \qquad (x, m) \mapsto x \cdot m$$

   such that

   (i) $\forall x \in X \,.\, x \cdot e = x$

[1] In the literature it is more common to consider *left* actions, but we prefer right ones for reasons that will become clear later in the Lecture.

(ii) $\forall x \in X\ \forall m, n \in M\,.\ (x \cdot m) \cdot n = x \cdot (mn)$

Given monoid actions $X \times M \to X$, $Y \times M \to Y$ on sets X, Y, an *equivariant map* between them is a function $f : X \to Y$ such that

$$\forall x \in X\ \forall m \in M\,.\ f(x \cdot m) = f(x) \cdot m.$$

Right M-actions and equivariant maps form a category M-**Set**.

(a) Show that M-**Set** has finite products

(b) Show that the multiplication map $M \times M \to M, (m, n) \to mn$ can be viewed as a right action of M on itself. To avoid confusion, we denote the corresponding object of M-**Set** by $\overline{M}$.

(c) Given $m \in M$ show that the function

$$\overline{m} : M \to M, \quad \overline{m}(n) = mn$$

is an equivariant map of type $\overline{M} \to \overline{M}$.

(d) Show that every equivariant map of type $\overline{M} \to \overline{M}$ is of the form $\overline{m}$ for some $m \in M$.

(e) Given $X, Y \in \mathrm{obj}(M\text{-}\mathbf{Set})$, show that the mapping

$$M\text{-}\mathbf{Set}(\overline{M} \times X, Y) \times M \to M\text{-}\mathbf{Set}(\overline{M} \times X, Y), \quad (h, m) \mapsto h \circ (\overline{m} \times \mathrm{id}_X)$$

defines a right M-action on M-**Set**$(\overline{M} \times X, Y)$.

Show that M-**Set**$(\overline{M} \times X, Y)$ equipped with this M-action is an exponential object Y^X in M-**Set**.

We conclude that M-**Set** is a cartesian closed category.