L108: Category theory and logic
Exercise sheet 3

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Revised: November 1, 2013, 18:19.

1. **Natural deduction**
   Write proofs in natural deduction of the judgments

   (a) $X \Rightarrow Y, Y \Rightarrow Z \vdash X \Rightarrow Z$
   (b) $Y \vdash (X \Rightarrow Y) \Rightarrow Y$
   (c) $X \vdash (X \Rightarrow Y) \Rightarrow Y$
   (d) $((X \Rightarrow Y) \Rightarrow Y) \Rightarrow Y \vdash X \Rightarrow Y$

   Write down the corresponding terms of simply typed lambda calculus.

   Solution. Here are the derivation trees:

$$\dfrac{\dfrac{\dfrac{X \Rightarrow Y, Y \Rightarrow Z, X \vdash X \qquad X \Rightarrow Y, Y \Rightarrow Z, X \vdash X \Rightarrow Y}{X \Rightarrow Y, Y \Rightarrow Z, X \vdash Y} \qquad X \Rightarrow Y, Y \Rightarrow Z, X \vdash Y \Rightarrow Z}{X \Rightarrow Y, Y \Rightarrow Z, X \vdash Z}}{X \Rightarrow Y, Y \Rightarrow Z \vdash X \Rightarrow Z}$$

$$\dfrac{Y, X \Rightarrow Y \vdash Y}{Y \vdash (X \Rightarrow Y) \Rightarrow Y}$$

$$\dfrac{\dfrac{X, X \Rightarrow Y \vdash X \Rightarrow Y \qquad X, X \Rightarrow Y \vdash X}{X, X \Rightarrow Y \vdash Y}}{X \vdash (X \Rightarrow Y) \Rightarrow Y}$$

$$\dfrac{\dfrac{A, X \vdash A \qquad \dfrac{\dfrac{A, X, X \Rightarrow Y \vdash X \Rightarrow Y \qquad A, X, X \Rightarrow Y \vdash X}{A, X, X \Rightarrow Y \vdash Y}}{A, X \vdash (X \Rightarrow Y) \Rightarrow Y}}{A, X \vdash Y}}{A \vdash X \Rightarrow Y}$$

   with $A \equiv ((X \Rightarrow Y) \Rightarrow Y) \Rightarrow Y$.

   And here are the terms:

   (a) $f{:}X \Rightarrow Y, g{:}Y \Rightarrow Z \vdash (\lambda x \,.\, g(fx)) : X \Rightarrow Z$
   (b) $y{:}Y \vdash (\lambda f \,.\, y) : (X \Rightarrow Y) \Rightarrow Y$

(c) $x{:}X \vdash (\lambda f\,.\,fx) : (X \Rightarrow Y) \Rightarrow Y$

(d) $x{:}((X \Rightarrow Y) \Rightarrow Y) \Rightarrow Y \vdash (\lambda y\,.\,x(\lambda z\,.\,zy)) : X \Rightarrow Y$

2. **Finite products**

(a) Define a concept of 'n-ary' product $A_1 \times \cdots \times A_n$ ($n \in \mathbb{N}$) of objects $A_1, \ldots, A_n \in \mathrm{obj}(\mathbb{C})$ of a category $\mathbb{C}$, generalizing the binary products introduced in the lecture.

Solution. An n-ary product of $A_1, \ldots, A_n \in \mathrm{obj}(\mathbb{C})$ is an object $A_1 \times \cdots \times A_n \in \mathrm{obj}(\mathbb{C})$ together with morphisms $\pi_i : A_1 \times \cdots \times A_n \to A_i$ for $1 \le i \le n$ such that for all $X \in \mathrm{obj}(\mathbb{C})$ and morphisms $f_i : X \to A_i$ ($1 \le i \le n$) there exists a unique $h : X \to A_1 \times \cdots \times A_n$ such that $\pi_i \circ h = f_i$ for all i with $1 \le i \le n$.

(b) For $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \mathrm{obj}(\mathbb{C})$, define a category $\mathbf{Span}_{\mathbb{C}}(A_1, \ldots, A_n)$ whose terminal objects are the n-ary products of $A_1, \ldots, A_n$.

Solution. The objects of $\mathbf{Span}_{\mathbb{C}}(A_1, \ldots, A_n)$ are tuples $(X, f_1, \ldots, f_n)$ with $f_i : X \to A_i$. Morphisms from $(X, f_1, \ldots, f_n)$ to $(Y, g_1, \ldots, g_n)$ are morphisms $h : X \to Y$ in $\mathbb{C}$ such that $g_i \circ h = f_i$ for $1 \le i \le n$.

(c) Show that if $\mathbb{C}$ has binary products and a terminal object, then all n-ary products exist.

Solution. The terminal object is a nullary product of the empty tuple. 'Unary' products are just given by the objects themselves. For $n \ge 3$, we deduce the existence of n-ary products from the existence of binary ones, by induction on n.
Assume that n-ary products and binary products exist in $\mathbb{C}$. Let $A_1, \ldots, A_{n+1} \in \mathbb{C}$. We define the $(n+1)$-ary product by $A_1 \times \cdots \times A_{n+1} = (A_a \times \cdots \times A_n) \times A_{n+1}$ (using the n-ary and the binary product), where the projection maps are the appropriate compositions of projections out of the n-ary and binary product.
Given X and $f_i : X \to A_i$ for $1 \le i \le n+1$, the $n+1$-fold target-tupeling is given by

$$\langle f_1, \ldots, f_{n+1} \rangle = \langle \langle f_1, \ldots, f_n \rangle, f_{n+1} \rangle.$$

Uniqueness follows from the respective uniqueness conditions for n-ary and binary products.

(d) Given $A_1, \ldots, A_n, B \in \mathrm{obj}(\mathbb{C})$, construct an isomorphism $j : (A_1 \times \cdots \times A_n) \times B \xrightarrow{\cong} A_1 \times \cdots \times A_n \times B$ (the left expression is a binary product of a of an n-ary product and a single object, while the right expression is an $(n+1)$-ary product).

Solution. In the proof of (c) we showed that $j : (A_1 \times \cdots \times A_n) \times B$ equipped with the obvious projection maps is an $n+1$-ary product. In (b) we showed that products are terminal objects in a category of spans. Thus, if $A_1 \times \cdots \times A_n \times B$ is another product, there is a unique isomorphism between them which is compatible with all the projections.

3. Show that the category **Preord** of preorders is cartesian closed, i.e. show that all finite products and exponential objects exist in **Preord**.

Solution. A terminal object is given by the one-element preorder. Given preorders $(D, \leq)$ and $(E, \leq)$, a binary product is given by $(D \times E, \leq)$ where the ordering is defined by $(d, e) \leq (d', e') :\Leftrightarrow d \leq d' \wedge e \leq e'$. To show that this defines a product, we have to show that the projection functions are monotone, and that $\langle f, g\rangle$ is monotone for monotone f, g, which is straightforward.

It remains to show the existence of exponential objects. We define $(E, \leq)^{(D,\leq)} = (\mathbf{Preord}((D, \leq), (E, \leq)), \leq)$ with $f \leq g :\Leftrightarrow \forall d \in D \,.\, fd \leq gd$. The evaluation $\varepsilon : (E, \leq)^{(D,\leq)} \times (D, \leq) \to (E, \leq)$ is defined by $(f, d) \mapsto f(d)$ as in **Set**. Given $f : (C, \leq) \times (D, \leq) \to (E, \leq)$, the map $\tilde{f} : (C, \leq) \to (E, \leq)^{(D,\leq)}$ is defined by $\tilde{f}(c)(d) = f(c, d)$ (also as in **Set**). To show that this defines an exponential, it remains to show that

- ε is monotone,
- $\tilde{f}$ is well defined, i.e. $\tilde{f}(c)$ is monotone for all $c \in C$, and
- $\tilde{f}$ is monotone w.r.t. the order on the exponential defined above,

which is all straightforward. Uniqueness of $\tilde{f}$ follows then from the analogous argument in **Set**.

4. **Right monoid actions**

Let $(M, \cdot, e)$ be a monoid. In the following we will write the multiplication in M as juxtaposition mn instead of $m \cdot n$.

A *right action*[1] of M on a set X (also called a *right M-action*) is a function

$$X \times M \to X, \qquad (x, m) \mapsto x \cdot m$$

such that

(i) $\forall x \in X \,.\, x \cdot e = x$

(ii) $\forall x \in X \; \forall m, n \in M \,.\, (x \cdot m) \cdot n = x \cdot (mn)$

Given monoid actions $X \times M \to X$, $Y \times M \to Y$ on sets X, Y, an *equivariant map* between them is a function $f : X \to Y$ such that

$$\forall x \in X \; \forall m \in M \,.\, f(x \cdot m) = f(x) \cdot m.$$

Right M-actions and equivariant maps form a category M-**Set**.

(a) Show that M-**Set** has finite products

(b) Show that the multiplication map $M \times M \to M, (m, n) \to mn$ can be viewed as a right action of M on itself. To avoid confusion, we denote the corresponding object of M-**Set** by $\overline{M}$.

(c) Given $m \in M$ show that the function

$$\overline{m} : M \to M, \quad \overline{m}(n) = mn$$

is an equivariant map of type $\overline{M} \to \overline{M}$.

[1] In the literature it is more common to consider *left* actions, but we prefer right ones for reasons that will become clear later in the lecture.

(d) Show that every equivariant map of type $\overline{M} \to \overline{M}$ is of the form $\overline{m}$ for some $m \in M$.

(e) Given $X, Y \in \mathrm{obj}(M\text{-}\mathbf{Set})$, show that the mapping

$$M\text{-}\mathbf{Set}(\overline{M} \times X, Y) \times M \to M\text{-}\mathbf{Set}(\overline{M} \times X, Y), \quad (h, m) \mapsto h \circ (\overline{m} \times \mathrm{id}_X)$$

defines a right M-action on $M\text{-}\mathbf{Set}(\overline{M} \times X, Y)$.

Show that $M\text{-}\mathbf{Set}(\overline{M} \times X, Y)$ equipped with this M-action is an exponential object Y^X in $M\text{-}\mathbf{Set}$.

We conclude that $M\text{-}\mathbf{Set}$ is a cartesian closed category.