L108: Category theory and logic
Exercise sheet 1

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1. Can you define a category with exactly one object and two morphisms? How many such categories are there? Hint: write a 'multiplication table' for the composition of morphisms.

2. Let **Span** be the category with three objects X, L, R, whose only non-identity arrows are $p : X \to L$ and $q : X \to R$.

   Let **CoSpan** be the category with three objects L, R, Y, whose only non-identity arrows are $i : L \to Y$ and $j : R \to Y$.

   (a) Complete the definitions of **Span** and **CoSpan** by giving the sets of morphisms between all pairs of objects, and defining the composition operations.

   (b) Draw pictures of the categories **Span** and **Cospan** including all morphisms.

   (c) Does any of the categories have a terminal object?

3. Let $\mathbb{C}$ be a category. A morphism $m : A \to B$ in $\mathbb{C}$ is called a *monomorphism*, if for every object $X \in \mathrm{obj}(\mathbb{C})$ and for every pair $f, g : X \to A$ of morphisms we have

$$m \circ f = m \circ g \qquad \text{implies} \qquad f = g.$$

   (a) Prove that every isomorphism is a monomorphism.

   (b) Prove that if $m : A \to B$ and $n : B \to C$ are monomorphisms, then their composition $n \circ m : A \to C$ is a monomorphism

   (c) Prove that if $m : A \to B$ and $h : B \to C$ are morphisms in $\mathbb{C}$, and $h \circ m$ is a monomorphism, then m is a monomorphism.

   (d) Characterize the monomorphisms in the category **Set**.

4. Let $\mathbb{C}$ be a category. The dual of $\mathbb{C}$ is a category $\mathbb{C}^{\mathsf{op}}$ with the same objects as $\mathbb{C}$ and where $\mathbb{C}^{\mathsf{op}}(A, B) = \mathbb{C}(B, A)$.

   (a) Complete the definition of the category $\mathbb{C}^{\mathsf{op}}$.

   (b) Give an example of a finite category which is not the same as its dual (up to isomorphism).

   (c) Give an example of a finite category that is the same as its dual.

5. **Duality principle.** Using the concept of opposite category, we can *dualise* definitions in category theory. For example

   - an *initial object* in $\mathbb{C}$ is a terminal object in $\mathbb{C}^{\mathsf{op}}$

- an *epimorphism* in $\mathbb{C}$ is a monomorphism in $\mathbb{C}^{\mathsf{op}}$

Tasks:

(a) Spell out the definitions of initial object and of epimorphism.

(b) Does the category **Set** of sets have any initial objects?

(c) Characterise the epimorphisms in the category **Set**.