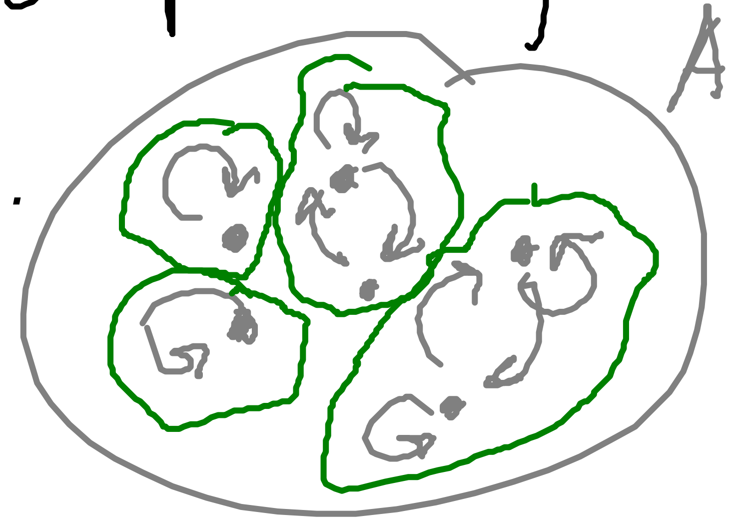
all partitions of A

**Theorem 105** *For every set $A$,*

$$\mathrm{EqRel}(A) \cong \mathrm{Part}(A) .$$

all equivalence relations on A



PROOF:

(1) $\mathrm{EqRel}(A) \to \mathrm{Part}(A)$

For an equivalence relation $E \subseteq A \times A$, define

$$\mathrm{part}(E) = \{ b \subseteq A \mid \exists a \in A .\ b = \underbrace{[a]_E}_{\text{the equivalence class of } a} \} = A/_E$$

notation

where $[a]_E = \{ x \in A \mid x \mathrel{E} a \}$

We need show $\mathrm{part}(E) = A/_E$ is a partition:

- $\emptyset \notin A/_E$ because $a \in [a]_E$
- $\bigcup_{b \in A/_E} b = A$
  $$\| \quad \bigcup_{a \in A} [a]_E$$
- $\forall b_1, b_2 \in A/_E .\ b_1 \neq b_2 \Rightarrow b_1 \cap b_2 = \emptyset$

Exercise

<u>Lemma</u>: $[a_1]_E = [a_2]_E \iff a_1 \, E \, a_2$

(2) Given a partition $P$ define an equivalence relation <u>equiv</u>$(P)$ on $A$.

$(a_1, a_2) \in$ <u>equiv</u>$(P)$ iff $\exists b \in P.\ a_1 \in b \ \& \ a_2 \in b$.

We need check that for all partitions $P$, <u>equiv</u>$(P)$ satisfies the 3 properties of equivalence relations.

REFLEXIVITY

SYMMETRY

TRANSITIVITY.

<u>Exercise.</u>

We have defined

$$\begin{tikzcd} \underline{\mathrm{EqRel}}(A) \arrow[r, bend left, "\underline{\mathrm{part}}"] & \underline{\mathrm{Part}}(A) \arrow[l, bend left, "\underline{\mathrm{equiv}}"] \end{tikzcd}$$

<u>Check</u>: $\underline{\mathrm{equiv}}(\underline{\mathrm{part}}(E)) = E$ <u>Exercise</u>.

$$\underline{\mathrm{part}}(\underline{\mathrm{equiv}}(P)) = P$$

# Calculus of bijections

the laws of arithmetic

$$a^{b+c} = a^b \cdot a^c$$

$$(B \uplus C) \Rightarrow A \cong (B \Rightarrow A) \times (C \Rightarrow A)$$

In ML $\quad \uplus \quad (\alpha, \beta)$ sum = one of $\alpha$ | two of $\beta$

$\times \quad *$

$\Rightarrow \quad \rightarrow$

# Characteristic (or indicator) functions

$$\mathcal{P}(\mathbf{A}) \cong (\mathbf{A} \Rightarrow [2]) \qquad [2] = \{0,1\}$$

- $\mathcal{P}(A) \longrightarrow (A \Rightarrow [2])$

  $S \longmapsto \left( a \in A \longmapsto \begin{cases} 0 & \text{if } a \notin S \\ 1 & \text{if } a \in S \end{cases} \right)$

Exercise: check bijective correspondence

<u>notation</u> $\chi_S$ — characteristic function of $S$

- $(A \Rightarrow [2]) \longrightarrow \mathcal{P}(A)$

  $f \longmapsto \{ x \in A \mid f(x) = 1 \} \subseteq A$

## Finite cardinality

$\{0, 1, \ldots, n-1\}$

$\parallel$

**Definition 106** *A set* $A$ *is said to be* finite *whenever* $A \cong [n]$ *for some* $n \in \mathbb{N}$*, in which case we write* $\#A = n$*.*

**Theorem 107** *For all* $m, n \in \mathbb{N}$,

1. $\mathcal{P}([n]) \cong [2^n]$
2. $[m] \times [n] \cong [m \cdot n]$
3. $[m] \uplus [n] \cong [m + n]$
4. $([m] \rightharpoonup [n]) \cong [(n + 1)^m]$
5. $([m] \Rightarrow [n]) \cong [n^m]$

They were an exam question last year.

Exercise.

Hint: For $n=2$, use some binary encoding.

# Infinity axiom

$\boxed{\text{There is an infinite set, containing } \emptyset \text{ and closed under successor.}}$

# Bijections

**Proposition 108** *For a function $f : A \to B$, the following are equivalent.*

1. $f$ *is bijective.*
2. $\forall\, b \in B.\, \exists!\, a \in A.\, f(a) = b$.
3. $\big(\forall\, b \in B.\, \exists\, a \in A.\, f(a) = b\big)$ Surjection

   &

   $\big(\forall\, a_1, a_2 \in A.\, f(a_1) = f(a_2) \implies a_1 = a_2\big)$

   Injection

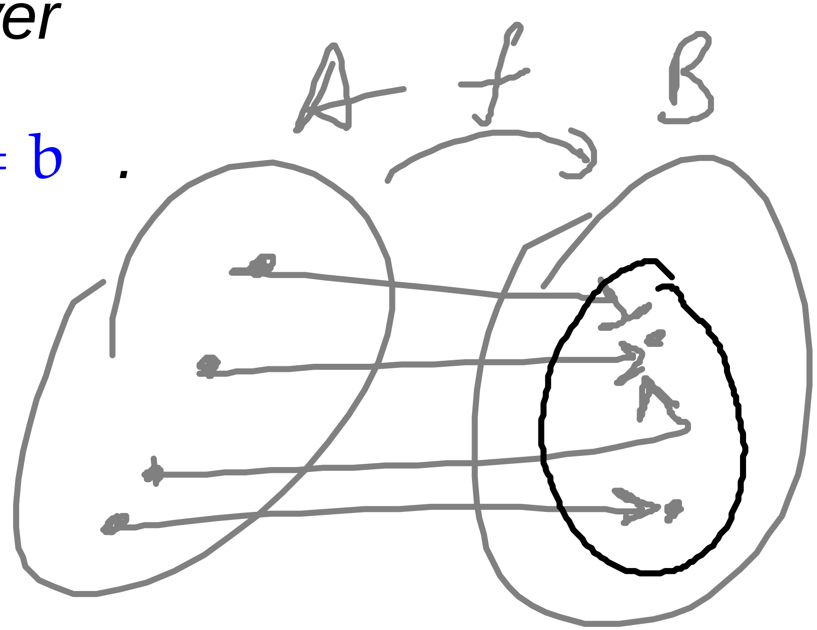
$$\vec{f}(A) = \{ b \in B \mid \exists a \in A.\ f(a) = b \}$$ — The direct image of A under f

## Surjections

**Definition 109** *A function* $f : A \to B$ *is said to be* surjective, *or a* surjection, *and indicated* $f : A \twoheadrightarrow B$ *whenever*

$$\forall b \in B.\ \exists a \in A.\ f(a) = b \quad .$$

equivalently $\vec{f}(A) = B$



**Theorem 110** *The identity function is a surjection, and the composition of surjections yields a surjection.*

The set of surjections from $A$ to $\mathrm{B}$ is denoted

$$\mathrm{Sur}(A, \mathrm{B})$$

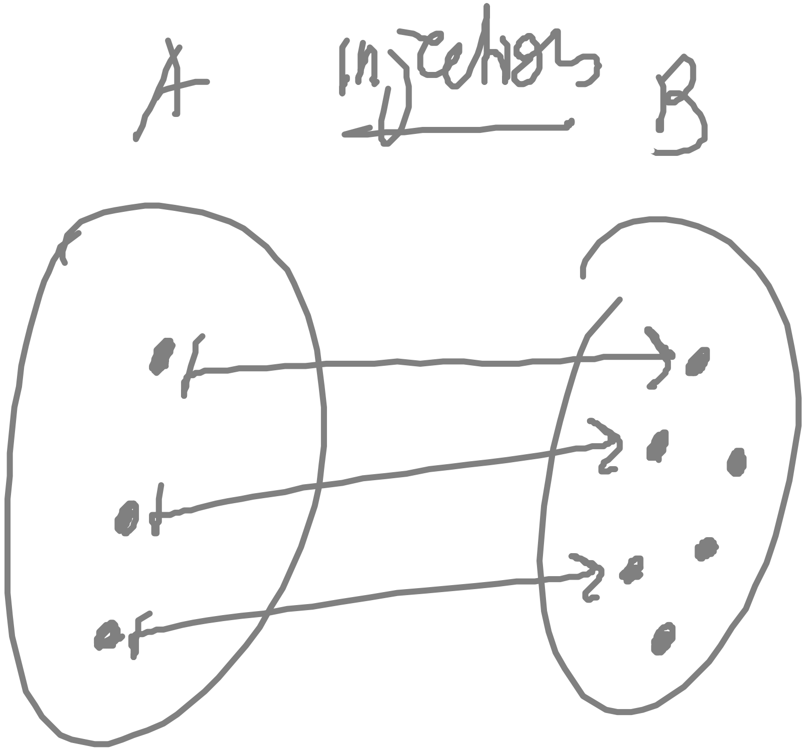
and we thus have

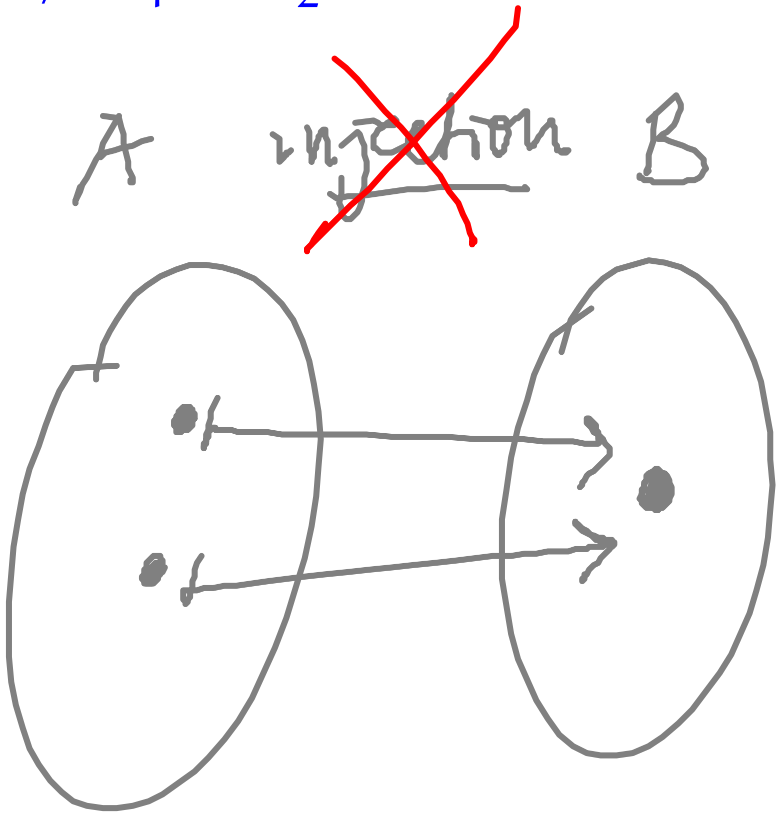
$$\mathrm{Bij}(A, \mathrm{B}) \subseteq \mathrm{Sur}(A, \mathrm{B}) \subseteq \mathrm{Fun}(A, \mathrm{B}) \subseteq \mathrm{PFun}(A, \mathrm{B}) \subseteq \mathrm{Rel}(A, \mathrm{B}) \ .$$

# Injections

**Definition 114** *A function* $f : A \to B$ *is said to be* injective*, or an* injection*, and indicated* $f : A \rightarrowtail B$ *whenever*

$$\forall a_1, a_2 \in A.\ \big(f(a_1) = f(a_2)\big) \implies a_1 = a_2 \quad .$$



The direct image $\overrightarrow{f}(A)$ of an injective function $f : A \to B$ is in bijection with $A$.

In symbols,

$$\overrightarrow{f}(A) \cong A.$$

**Theorem 115** *The identity function is an injection, and the composition of injections yields an injection.*

The set of injections from $A$ to $B$ is denoted

$$\mathrm{Inj}(A, B)$$

and we thus have

$$\begin{tikzcd}
 & \mathrm{Sur}(A, B) \arrow[dr, phantom, sloped, "\subseteq"] & \\
\mathrm{Bij}(A, B) \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \mathrm{Fun}(A, B) \subseteq \mathrm{PFun}(A, B) \subseteq \mathrm{Rel}(A, B) \\
 & \mathrm{Inj}(A, B) \arrow[ur, phantom, sloped, "\subseteq"] &
\end{tikzcd}$$

with

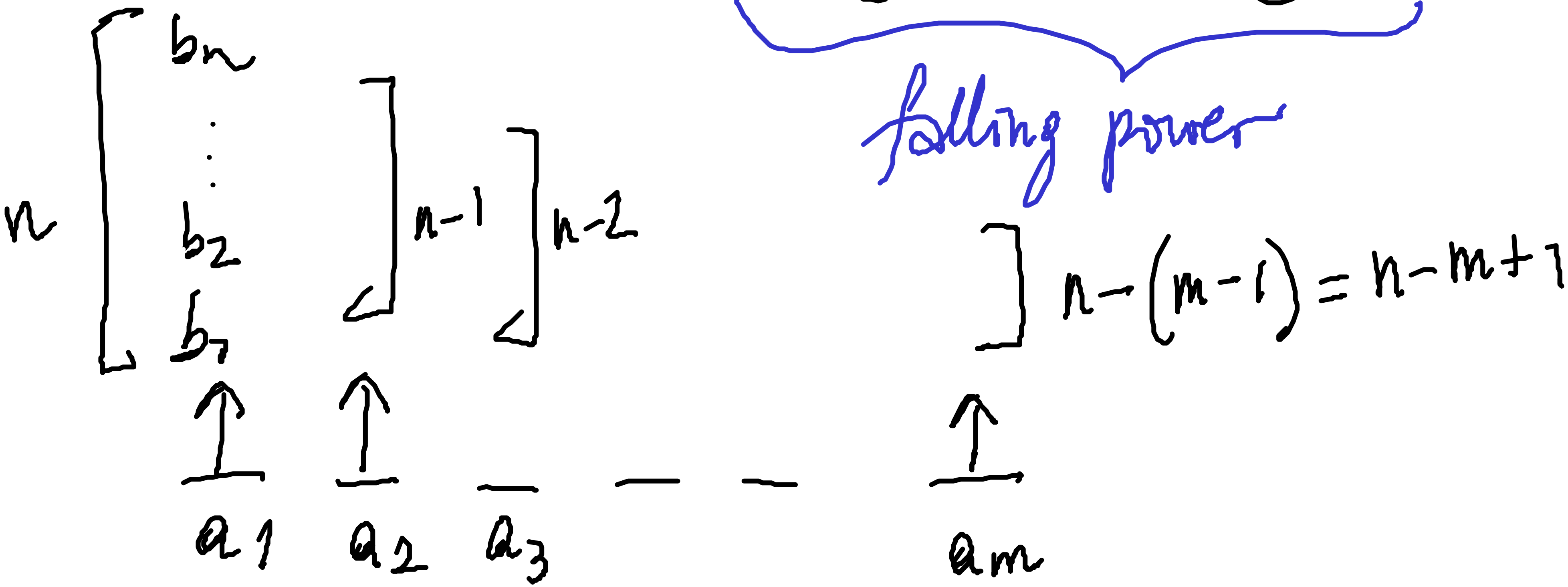
$$\mathrm{Bij}(A, B) = \mathrm{Sur}(A, B) \cap \mathrm{Inj}(A, B) \quad .$$

**Proposition 116** *For all finite sets* $A$ *and* $B$,

$$\#\mathrm{Inj}(A,B) = \begin{cases} \binom{\#B}{\#A}\cdot(\#A)! & \text{, if } \#A \le \#B \\ 0 & \text{, otherwise} \end{cases}$$

PROOF IDEA:

In Total $\underbrace{n\cdot(n-1)\cdot\;\ldots\;\cdot(n-m+1)}_{\text{falling power}}$

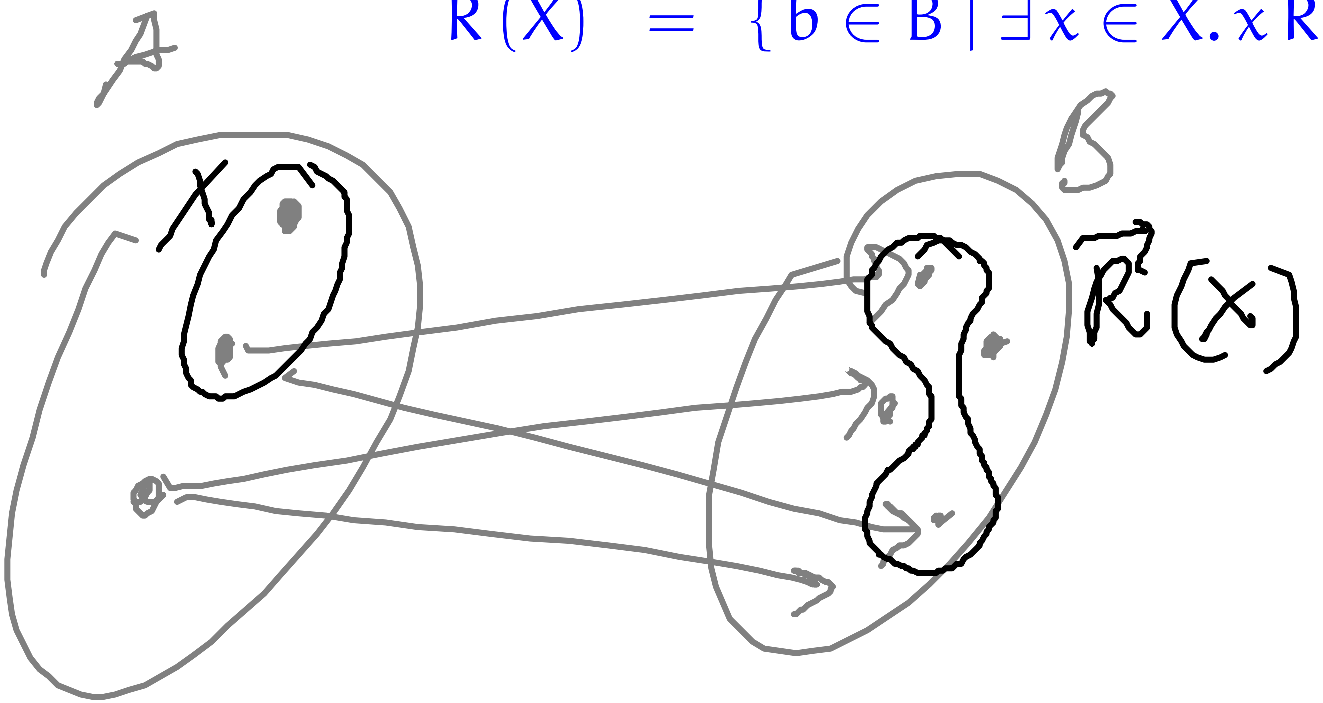


# Images

**Definition 117** *Let* $R : A \longrightarrow\!\!\!\!\!\!\!+\;\; B$ *be a relation.*

- *The* direct image *of* $X \subseteq A$ *under* $R$ *is the set* $\overrightarrow{R}(X) \subseteq B$, *defined as*

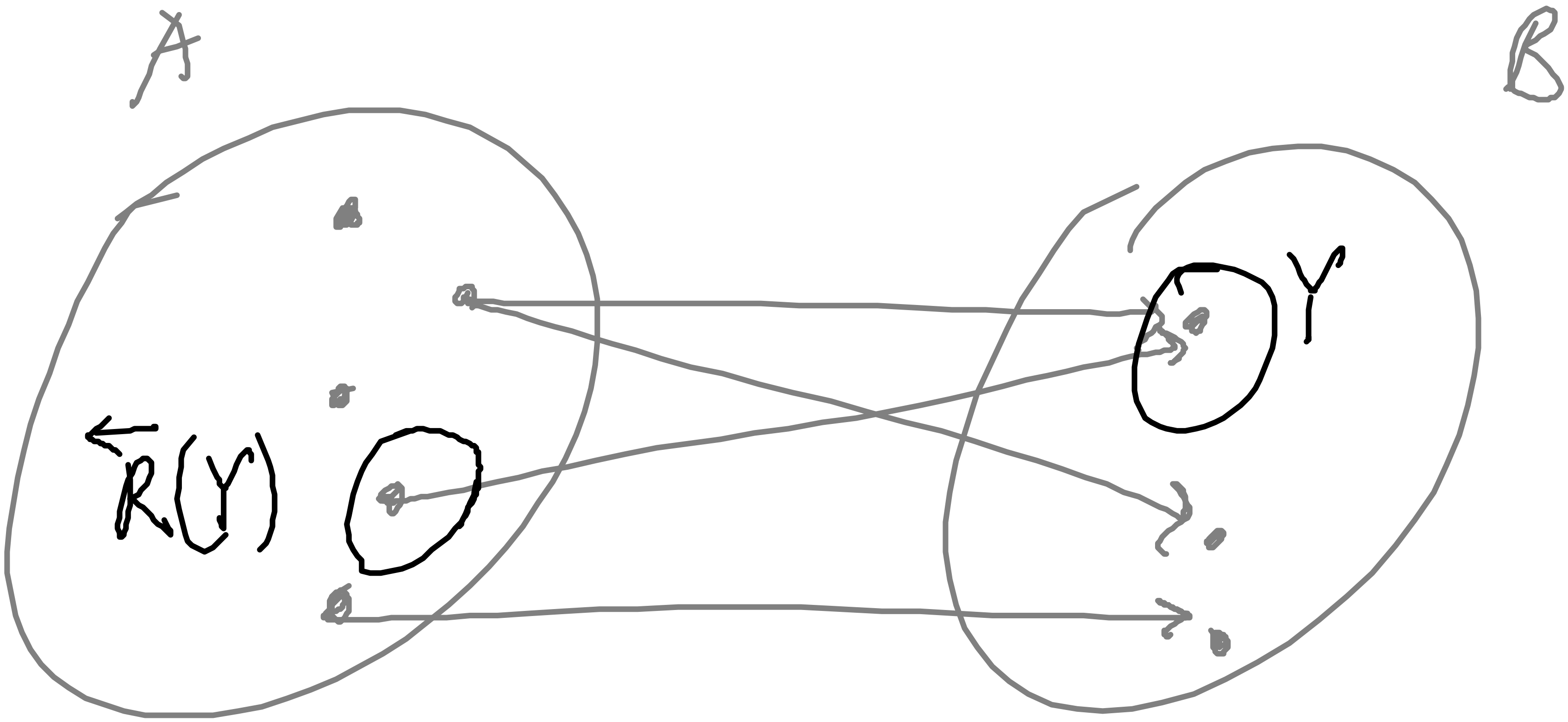
$$\overrightarrow{R}(X) \;=\; \{\, b \in B \mid \exists x \in X.\, x \mathrel{R} b \,\} \;.$$




***NB*** *This construction yields a function* $\overrightarrow{R} : \mathcal{P}(A) \to \mathcal{P}(B)$.

- *The* inverse image *of* $Y \subseteq B$ *under* $R$ *is the set* $\overleftarrow{R}(Y) \subseteq A$, *defined as*

$$\overleftarrow{R}(Y) \;=\; \{\, a \in A \mid \forall\, b \in B.\, a\,R\,b \implies b \in Y \,\}$$




***NB*** *This construction yields a function* $\overleftarrow{R} : \mathcal{P}(B) \to \mathcal{P}(A)$.

# Replacement axiom

The direct image of every definable functional property on a set is a set.

$$i \longmapsto A_i$$

$$\{A_i \mid i \in I\}$$

a set.

# Set-indexed constructions

For every mapping associating a set $A_i$ to each element of a set $I$, we have the set

$$\bigcup_{i \in I} A_i \;=\; \bigcup \{ A_i \mid i \in I \} \;=\; \{ a \mid \exists\, i \in I.\, a \in A_i \} \quad .$$

**Examples:**

1. Indexed disjoint unions:

$$\biguplus_{i \in I} A_i \;=\; \bigcup_{i \in I} \{i\} \times A_i$$

2. Finite sequences on a set $A$:

$$A^* \;=\; \biguplus_{n \in \mathbb{N}} A^n$$