*correction

Example: The following defines a partial function $\mathbb{Z} \times \mathbb{Z} \rightharpoonup \mathbb{Z} \times \mathbb{Z}$:

- for $n \geq 0$ and $m > 0$,
  $(n, m) \mapsto \big(\,\mathrm{quo}(n, m)\,,\ \mathrm{rem}(n, m)\,\big)$
- for $n \geq 0$ and $m < 0$,
  $(n, m) \mapsto \big(\,-\mathrm{quo}(n, -m)\,,\ \mathrm{rem}(n, -m)\,\big)$
- * for $n < 0$ and $m > 0$,
  $(n, m) \mapsto \big(\,-\mathrm{quo}(-n, m) - 1\,,\ \mathrm{rem}(m - \mathrm{rem}(-n, m), m)\,\big)$
- * for $n < 0$ and $m < 0$,
  $(n, m) \mapsto \big(\,\mathrm{quo}(-n, -m) + 1\,,\ \mathrm{rem}(-m - \mathrm{rem}(-n, -m), -m)\,\big)$

Its domain of definition is $\{\,(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid m \neq 0\,\}$.

Functions (or maps)

Definition 97 *A partial function is said to be* total*, and referred to as a* (total) function *or* map*, whenever its domain of definition coincides with its source.*

$$\underline{Fun}(A,B) \subseteq \underline{PFun}(A,B) \subseteq \underline{Rel}(A,B)$$

$$Fun(A,B) = (A \Rightarrow B) \qquad PFun(A,B) = (A \rightharpoonup B) \qquad Rel(A,B) = \mathcal{P}(A \times B)$$

$$\#B^{\#A} \qquad (\#B+1)^{\#A} \qquad 2^{\#A \cdot \#B}$$

Theorem 98 *For all* $f \in \mathrm{Rel}(A, B)$,

$$f \in (A \Rightarrow B) \iff \forall\, a \in A.\, \exists!\, b \in B.\ a\, f\, b \quad .$$

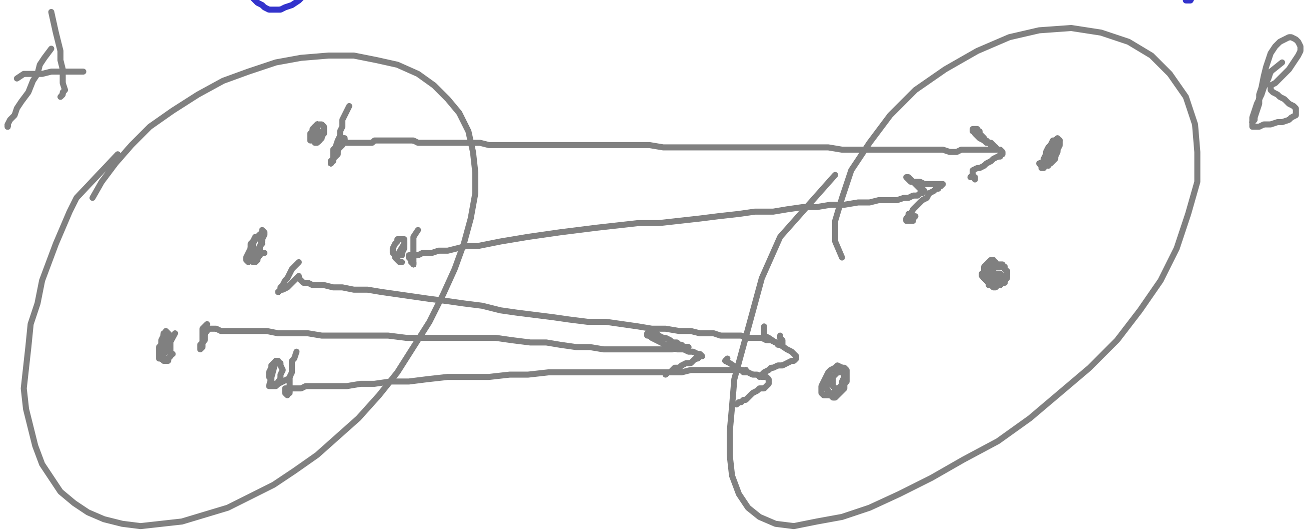
PROOF: For all $a \in A$,

TOTAL

1 there is a $b \in B$ s.t. $a\,f\,b$

FUNCTIONAL

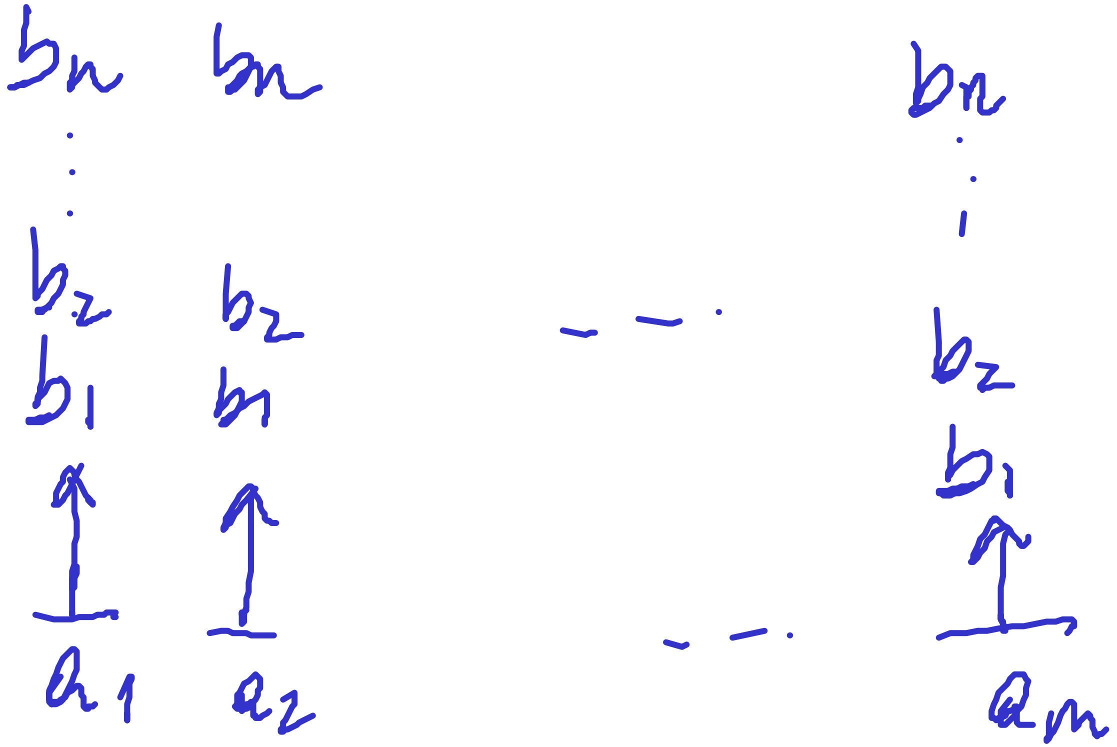
2 such a b is unique for each a



Proposition 99 *For all finite sets* A *and* B,

$$\#(A \Rightarrow B) = \#B^{\#A} \quad .$$

PROOF IDEA:



$$n \cdot n \cdot \ldots \cdot n = n^m$$

Say $f, g : A \to B$, Then $f = g$ iff $\forall a \in A.\ f(a) = g(a)$

$\boxed{\text{Function extensionality}}$

Theorem 100 *The identity partial function is a function, and the composition of functions yields a function.*

NB For all sets A, the identity function $\mathrm{id}_A : A \to A$ is given by the rule

$$\mathrm{id}_A(a) = a$$

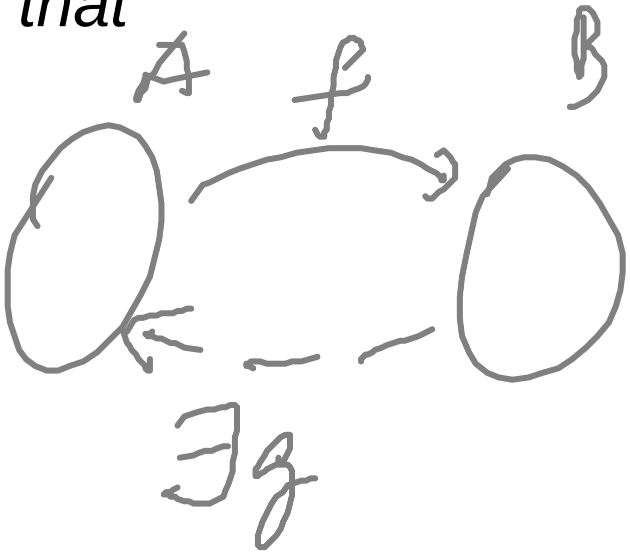
and, for all functions $f : A \to B$ and $g : B \to C$, the composition function $g \circ f : A \to C$ is given by the rule

$$(g \circ f)(a) = g\big(f(a)\big) \quad .$$

$\underline{Bij}(A,B) \subseteq \underline{Fun}(A,B) \subseteq \underline{PFun}(A,B) \subseteq \underline{Rel}(A,B)$

Bijections

Definition 101 *A function* $f : A \to B$ *is said to be* bijective, *or a* bijection, *whenever there exists a (necessarily unique) function* $g : B \to A$ *(referred to as the* inverse *of* f*) such that*



1. g *is a* left inverse *for (or a* retraction *of)* f*:*

$$g \circ f = \mathrm{id}_A \quad ,$$

$\forall a \in A.\ g(f\,a) = a$

2. g *is a* right inverse *for (or a* section *of)* f*:*

$$f \circ g = \mathrm{id}_B \quad .$$

$\forall b \in B.\ f(g\,b) = b$

$$\{R \subseteq [n]\times[m]\} \cong \{(n\times m)\text{-Matrices}\}$$

$$R \longmapsto \underline{\mathrm{mat}}(R)$$

$$\underline{\mathrm{rel}}(M) \longleftarrow\!\!\shortmid\; M$$

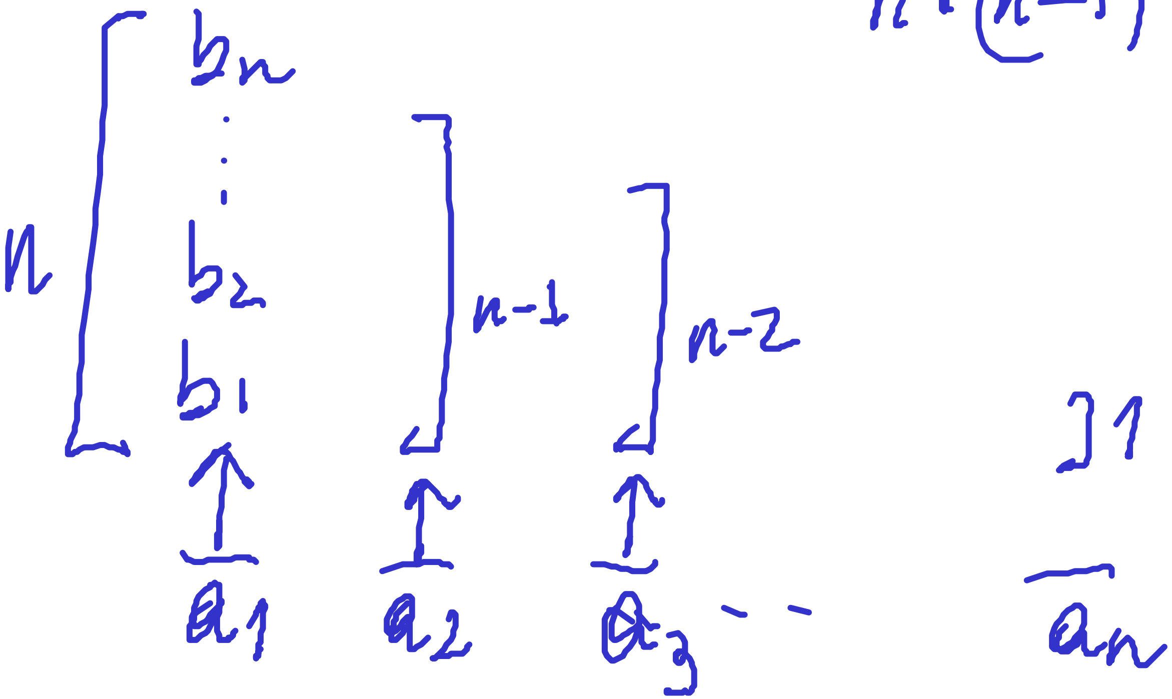
Isomorphism
(or same cardinality).

Proposition 102 *For all finite sets A and B,*

$$\#\,\mathrm{Bij}(A,B) \;=\; \begin{cases} 0 & \text{, if } \#A \neq \#B \\ n! & \text{, if } \#A = \#B = n \end{cases}$$

PROOF IDEA:

$$n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 1 = n!$$



Theorem 103 *The identity function is a bijection, and the composition of bijections yields a bijection.*

$$\begin{tikzcd} A \arrow[r, bend left, "id_A"] & A \arrow[l, bend left, "id"] \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[rr, bend left=60, "g \circ f"] & B \arrow[l, bend left, "f^{-1}"] \arrow[r, bend left, "g"] & C \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

$f^{-1} \circ f = id_A$

$f \circ f^{-1} = id_B$

<u>claim</u> $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

$g^{-1} \circ g = id_B$

$g \circ g^{-1} = id_C$

__Check__

① $(g \circ f) \circ (f^{-1} \circ g^{-1}) = Id$

② $(f^{-1} \circ g^{-1}) \circ (g \circ f) = Id$

For ① $(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1}$

$$= g \circ Id \circ g^{-1}$$

$$= g \circ g^{-1}$$

$$= Id.$$

$$n \longmapsto n+1$$
$$\mathbb{N} \rightleftarrows \mathbb{N}^+$$
$$k-1 \longleftarrow\!\shortmid k$$

Definition 104 *Two sets* A *and* B *are said to be* isomorphic *(and to have the* same cardinatity*) whenever there is a bijection between them; in which case we write*

$$A \cong B \quad \text{or} \quad \#A = \#B \quad .$$

Examples:

1. $\{0, 1\} \cong \{\mathbf{false}, \mathbf{true}\}$.

2. $\mathbb{N} \cong \mathbb{N}^+$, $\mathbb{N} \cong \mathbb{Z}$, $\mathbb{N} \cong \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \cong \mathbb{Q}$.

i) $\mathbb{N} \supseteq \{n \in \mathbb{N} \mid n > 0\} = \mathbb{N}^+$

Equivalence relations and set partitions

- Equivalence relations. (on a set, say A)

$E \subseteq A \times A$

Reflexive

① $\forall a \in A,\ aEa$

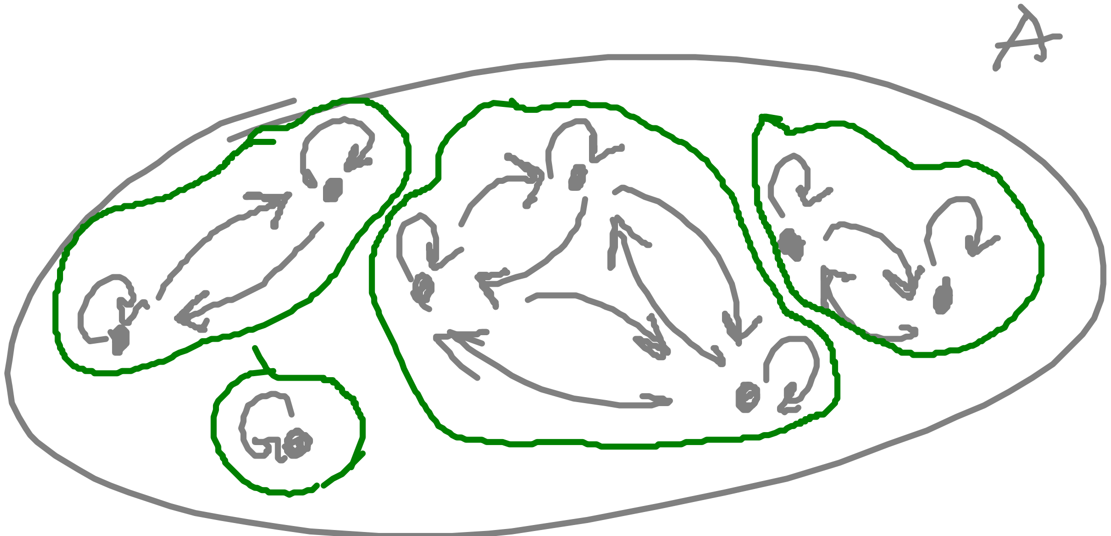
Transitive

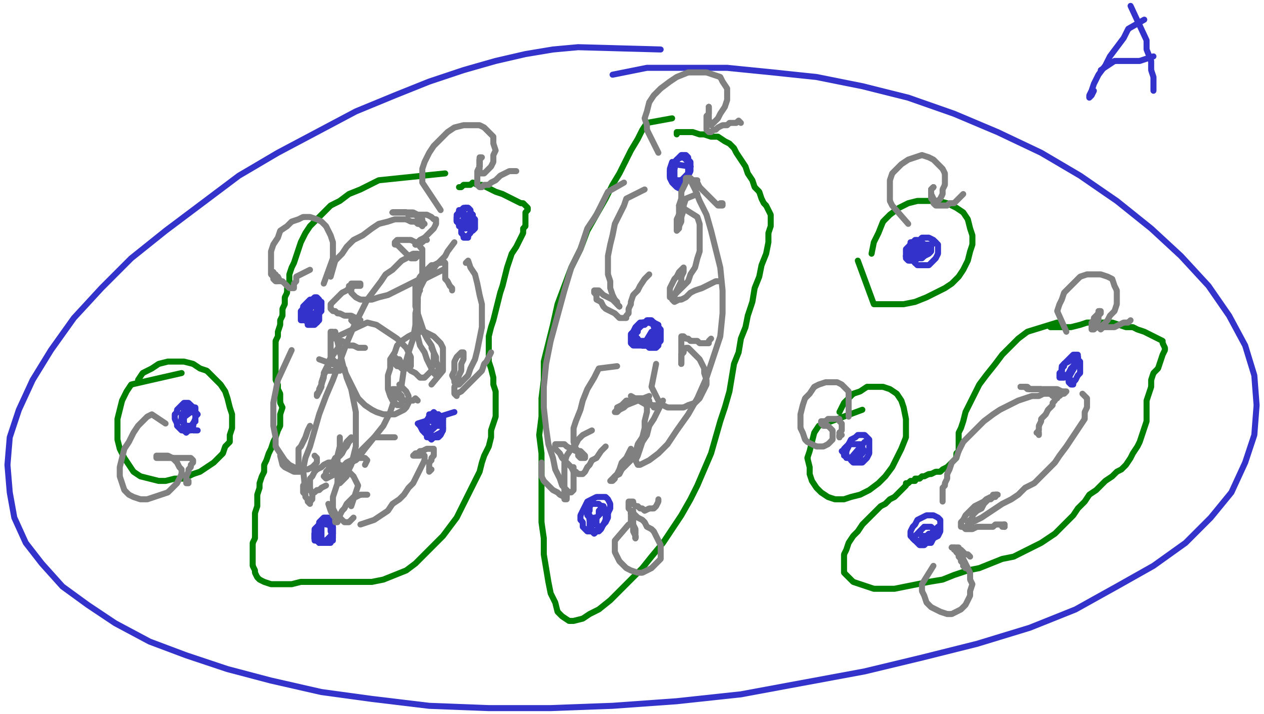
② $\forall a,b,c \in A,\ aEb \ \& \ bEc \Rightarrow aEc$

SYMMETRIC

③ $\forall a,b \in A,\ aEb \Rightarrow bEa$

Every equivalence relation yields a partition on the set.





a partition
(of six blocks)
induces an
equivalence
relation

- Set partitions.

$$P \subseteq \mathcal{P}(A)$$

1. $\emptyset \notin P$

2. $\bigcup P = \bigcup_{b \in P} b = A$

3. $\forall b_1, b_2 \in P.\ b_1 \neq b_2 \Rightarrow b_1 \cap b_2 = \emptyset$

Theorem 105 *For every set* A*,*

$$\mathrm{EqRel}(A) \cong \mathrm{Part}(A) \quad .$$

PROOF:

The set of all equivalence relation on A

The set of all The partitions of A