For $A, B \in \mathcal{P}(\mathcal{U})$, $A \cup B = \{ x \in \mathcal{U} \mid x \in A \vee x \in B \}$

$A \cap B = \{ x \in \mathcal{U} \mid x \in A \;\&\; x \in B \}$

Sets and logic

$\mathcal{P}(\mathrm{U})$	$\{$ **false** , **true** $\}$
$\emptyset$	**false**
U	**true**
$\cup$	$\vee$
$\cap$	&
$(\cdot)^{c}$	$\neg(\cdot)$

$\mathcal{P}(\{*\}) = \{ \emptyset, \{*\} \}$

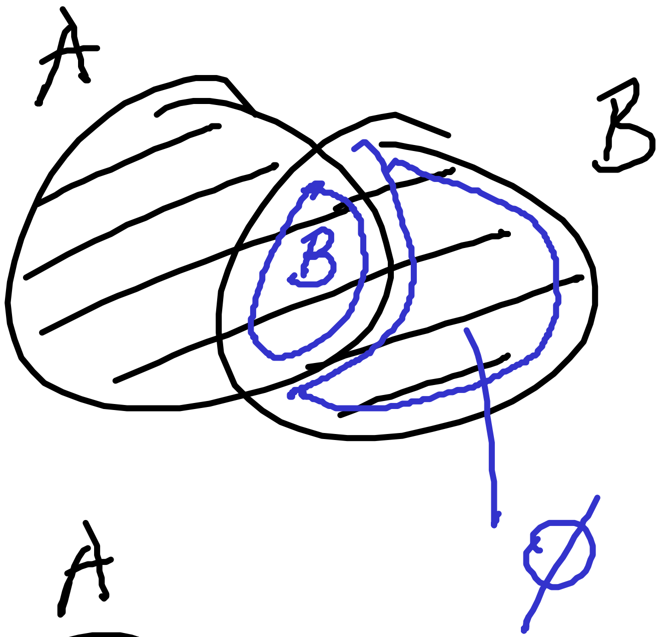
false true

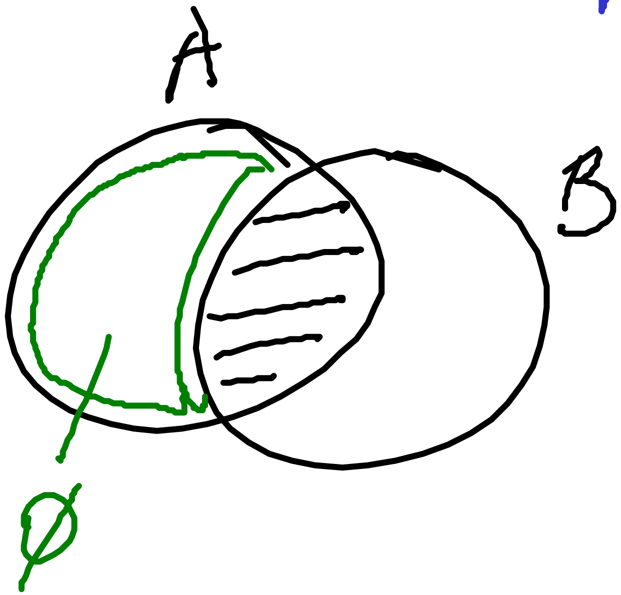
$A \cup B = A$

$\Longleftrightarrow B \subseteq A$

$A \cap B = A$

$\Longleftrightarrow A \subseteq B$





$X \cup (X \cap A) = X$ corresponds $X \cap A \subseteq X$

$X \cap (X \cup A) = X$ corresponds $X \subseteq X \cup A$

Unordered

Pairing axiom

For every a and b, there is a set with a and b as its only elements.

$a = b \Rightarrow$ #1

$\{\, a \,,\, b \,\}$

$a \neq b \Rightarrow$ #2

defined by

$$\forall x.\, x \in \{a, b\} \iff (x = a \vee x = b)$$

$\{a, b\} = \{b, a\}$

NB The set $\{a, a\}$ is abbreviated as $\{\, a \,\}$, and referred to as a *singleton*.

for $a \neq b$

$\langle a, b \rangle \neq \langle b, a \rangle$

Ordered pairing

For every pair a and b, the set

$$\{\{a\}, \{a, b\}\}$$

is abbreviated as

$$\langle a, b \rangle$$

and referred to as an *ordered pair*.

<u>Exercise</u> Find other encodings for ordered pairing.

Proposition 71 (Fundamental property of ordered pairing)

For all a, b, x, y,

$$\langle a, b\rangle = \langle x, y\rangle \iff (a = x \;\&\; b = y) \quad .$$

PROOF: $(\Longleftarrow)$ ✓

$(\Longrightarrow)$ $\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\}$

<u>Case 1</u>: $a = b \Longrightarrow \{\{a\},\{a,b\}\} = \{\{a\}\} = \{\{x\},\{x,y\}\}$

$$\Downarrow$$

$$\{x\} = \{x,y\}$$

$$\Downarrow$$

$$\{\{a\}\} = \{\{x\}\} = \{\{x\},\{x,y\}\} \Longleftarrow x = y$$

$$\Uparrow$$

$$\{a\} = \{x\} \Longrightarrow a = x$$

Case 2 $a \neq b$

EXERCISE

In ML, the * type constructor.

Products

The *product* $A \times B$ of two sets A and B is the set

$$A \times B = \{\, x \mid \exists\, a \in A, b \in B.\, x = (a, b)\,\}$$

an ordered pairing construction

where

$$\forall a_1, a_2 \in A, b_1, b_2 \in B.$$
$$(a_1, b_1) = (a_2, b_2) \iff (a_1 = a_2 \;\&\; b_1 = b_2) \quad .$$

Thus,

$$\forall\, x \in A \times B.\, \exists!\, a \in A.\, \exists!\, b \in B.\, x = (a, b) \quad .$$

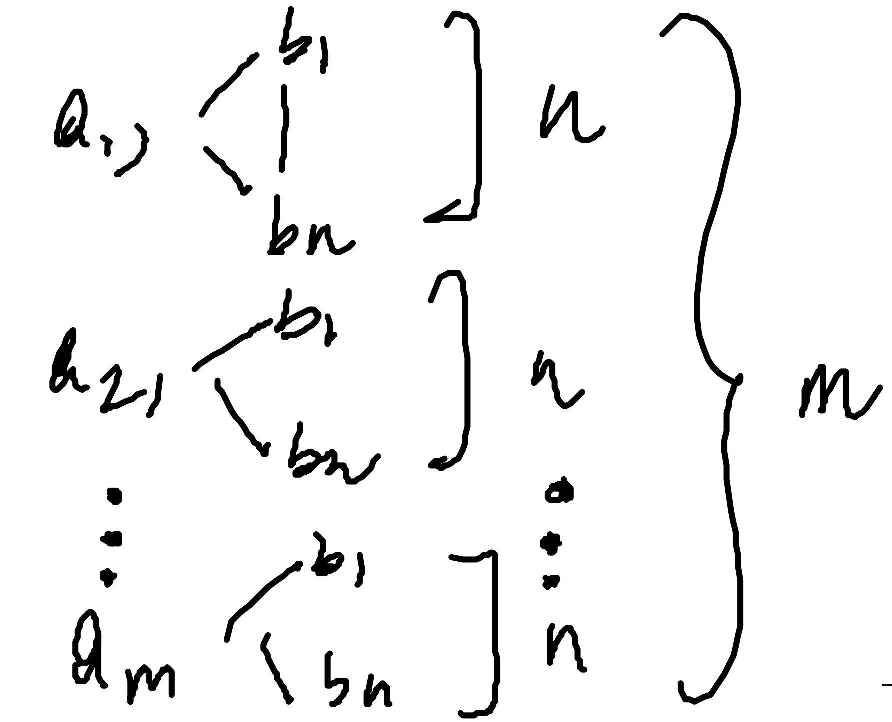
Proposition 73 *For all finite sets* A *and* B,

$$\#(A \times B) = \#A \cdot \#B \quad .$$

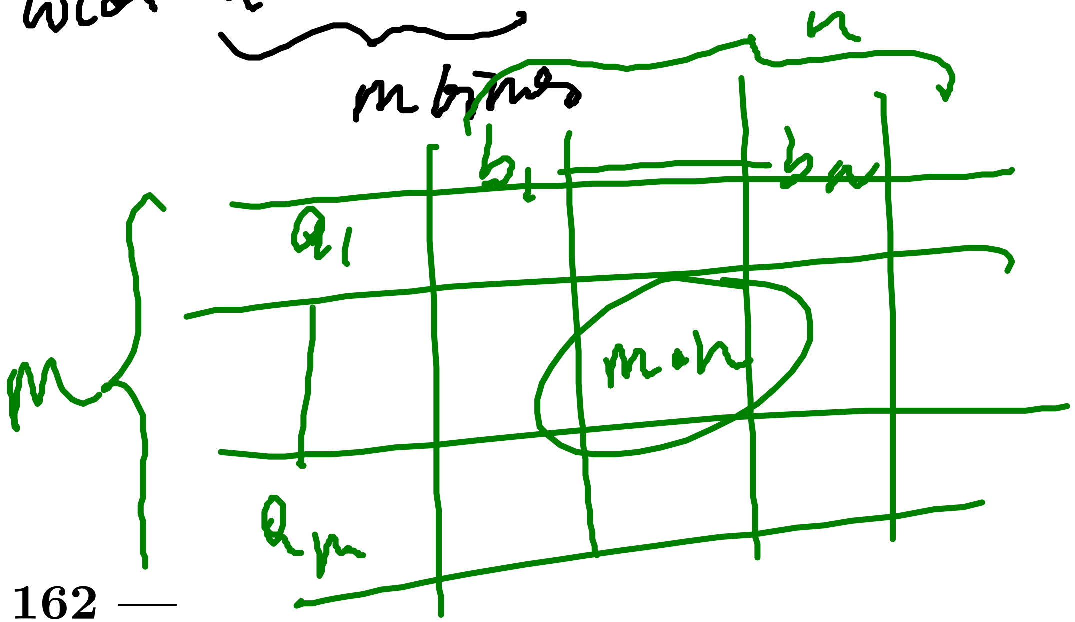
PROOF IDEA: $A = \{a_1, \ldots, a_m\}$ $(\#A = m)$

$B = \{b_1, \cdots, b_n\}$ $(\#B = n)$

$A \times B = \{(a_i, b_j) \mid i = 1 - m, \ j = 1 - n\}$



total $\underbrace{n + \cdots + n}_{m \text{ times}} = m \cdot n$



Big unions and intersections

Given A and B we considered $A \cup B$

Given $A_1, A_2, \ldots A_n$ we have $A_1 \cup A_2 \cup \cdots \cup A_n$

Consider $\mathcal{F} \subseteq P(\mathcal{U})$

{a set of subsets of $\mathcal{U}$

and define $\bigcup \mathcal{F} = \{x \in \mathcal{U} \mid \exists X \in \mathcal{F}.\, x \in X\}$

notation $\displaystyle\bigcup_{X \in \mathcal{F}} X$

example $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$

$\bigcup \mathcal{F} = A_1 \cup A_2 \cup \cdots \cup A_n$

Intuition: In ML

flatten : α list list $\to$ α list

$[L_1, \ldots, L_n] \longmapsto L_1 @ \ldots @ L_n$

Given $\mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$

Consider

$$\bigcap \mathcal{F} = \{ x \in \mathcal{U} \mid \forall X \in \mathcal{F}.\ x \in X \}$$

$$\bigcap \{A_1, \ldots, A_n\} = A_1 \cap \ldots \cap A_n$$

a closure property

Theorem 74 *Let*

$$\mathcal{F} = \left\{ S \subseteq \mathbb{R} \;\middle|\; (0 \in S) \;\&\; (\forall x \in \mathbb{R}.\, x \in S \implies (x+1) \in S) \right\} .$$

Then, (i) $\mathbb{N} \in \mathcal{F}$ *and* (ii) $\mathbb{N} \subseteq \bigcap \mathcal{F}$. *Hence,* $\bigcap \mathcal{F} = \mathbb{N}$.

Example

$\mathbb{R} \in \mathcal{F}$

$\mathbb{Q} \in \mathcal{F}$

$\mathbb{Z} \in \mathcal{F}$

$\{0\} \notin \mathcal{F}$

$\emptyset \notin \mathcal{F}$

PROOF:

(ii) $\forall n \in \mathbb{N}.\ n \in \bigcap \mathcal{F}$

$\iff \forall n \in \mathbb{N}.\ \boxed{\forall S \in \mathcal{F}.\ n \in S}$

$\Updownarrow$ $P(n)$

Prove $\forall n \in \mathbb{N}.\ P(n)$ by induction.

Exercise

$\mathbb{N}$ is the least set of numbers containing 0 and closed under successor.

For $\mathcal{F} \subseteq \mathcal{P}(\mathcal{P}(\mathcal{U}))$,

$$\bigcup\left(\bigcup \mathcal{F}\right) = \bigcup\left\{\bigcup \mathcal{A} \mid \mathcal{A} \in \mathcal{F}\right\}$$

or, in indexed notation,

$$\bigcup_{X \in \left(\bigcup_{\mathcal{A} \in \mathcal{F}} \mathcal{A}\right)} X = \bigcup_{\mathcal{A} \in \mathcal{F}} \left(\bigcup_{X \in \mathcal{A}} X\right)$$

IDEA:

$\mathcal{F} = \{ -, \mathcal{A}, - \}$ $\mathcal{A} = \{ -, X, - \}$

$\bigcup \mathcal{F} = - \cup \mathcal{A} \cup -$

$= \{ -, X, - \}$

$\bigcup (\bigcup \mathcal{F}) = - \cup X \cup -$

$\bigcup \mathcal{A} = (- \cup X \cup -)$

$\bigcup \{ \bigcup \mathcal{A} \mid \mathcal{A} \in \mathcal{F} \}$

$= - \cup (- \cup X \cup -) \cup -$

PROOF: $x \in \bigcup_{X \in \left(\bigcup_{\mathcal{A} \in \mathcal{F}} \mathcal{A}\right)} X$

$\iff \exists X.\ X \in \left(\bigcup_{\mathcal{A} \in \mathcal{F}} \mathcal{A}\right)\ \&\ x \in X$

$\iff \exists X.\ \exists \mathcal{A}.\ \mathcal{A} \in \mathcal{F}\ \&\ X \in \mathcal{A}\ \&\ x \in X$

$\iff \exists \mathcal{A}.\ \mathcal{A} \in \mathcal{F}\ \&\ \exists X.\ X \in \mathcal{A}\ \&\ x \in X$

$\iff \exists \mathcal{A}.\ \mathcal{A} \in \mathcal{F}\ \&\ x \in \bigcup_{X \in \mathcal{A}} X$

$\iff x \in \bigcup_{\mathcal{A} \in \mathcal{F}} \left(\bigcup_{X \in \mathcal{A}} X \right)$

Compare the previous identity with the following one for lists.

For $L : \alpha$ list list list,

$$\text{flatten}(\text{flatten } L) = \text{flatten}(\text{map flatten } L)$$

Btw, this is one of the laws of a mathematical structure called a MONAD, which has become a fundamental notion in functional programming.

notation: $\bigcup \{A, B\} = A \cup B$

Union axiom

$\boxed{\text{Every collection of sets has a union.}}$

$$\bigcup \mathcal{F}$$

$$x \in \bigcup \mathcal{F} \iff \exists\, X \in \mathcal{F}.\, x \in X$$

For *non-empty* $\mathcal{F}$ we also have

$$\bigcap \mathcal{F}$$

defined by

$$\forall x.\ x \in \bigcap \mathcal{F} \iff \big(\forall X \in \mathcal{F}.\, x \in X\big) \quad .$$