Negation

Negations are statements of the form

$$\boxed{\text{not } P}$$

or, in other words,

$$\boxed{P \text{ is not the case}}$$

or

$$\boxed{P \text{ is absurd}}$$

or

$$\boxed{P \text{ leads to contradiction}}$$

or, in symbols,

$$\boxed{\neg P}$$

A first proof strategy for negated goals and assumptions:

If possible, reexpress the negation in an *equivalent* form and use instead this other statement.

$P \Rightarrow \neg(\neg P)$

Logical equivalences

$\neg(\neg P) \Rightarrow P$?

$P \Rightarrow (\neg P \Rightarrow false)$

Assume P (1)

Assume $\neg P$

$\Updownarrow$

(2) $(P \Rightarrow false)$

By MP, (1) & (2)

false.

$$
\begin{array}{rcl}
\neg(P \Longrightarrow Q) & \Longleftrightarrow & P \;\&\; \neg Q \\
\neg(P \Longleftrightarrow Q) & \Longleftrightarrow & \neg P \Longleftrightarrow \neg Q \\
\neg(\forall x.\, P(x)) & \Longleftrightarrow & \exists x.\, \neg P(x) \\
\neg(P \;\&\; Q) & \Longleftrightarrow & (\neg P) \vee (\neg Q) \\
\neg(\exists x.\, P(x)) & \Longleftrightarrow & \forall x.\, \neg P(x) \\
\neg(P \vee Q) & \Longleftrightarrow & (\neg P) \;\&\; (\neg Q) \\
\neg(\neg P) & \Longleftrightarrow & P \\
\neg P & \Longleftrightarrow & (P \Rightarrow \mathbf{false})
\end{array}
$$

Truth tables

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$$(P \Rightarrow Q) \Longleftrightarrow (\neg P \vee Q)$$

$$P \& \neg Q$$

$$\neg (P \Rightarrow Q) \Leftrightarrow \neg (\neg P \vee Q) \Leftrightarrow \neg\neg P \& \neg Q$$

Theorem 33 *For all statements* P *and* Q,

$$(P \implies Q) \implies (\neg Q \implies \neg P) .$$

PROOF: Let P and Q be statements. Assume ① $\boxed{P \Rightarrow Q}$. Assume $\neg Q \iff \boxed{(Q \Rightarrow \text{false})}$ ②

By MP, $P \Rightarrow Q \Rightarrow \text{false}$

Therefore $(P \Rightarrow \text{false}) \iff \neg P$. □

$\boxed{?}$ $(\neg Q \Rightarrow \neg P) \Rightarrow (P \Rightarrow Q)$?

Theorem 34 *The real number* $\sqrt{2}$ *is irrational.*

PROOF:

$\neg(\sqrt{2}$ rational$)$

$\iff (\sqrt{2}$ rational $\Rightarrow$ false$)$

Assume $\sqrt{2}$ rational

$\iff (\exists$ int $m, n.\ \sqrt{2} = m/n)$

Let m_0 and n_0 be integers such that $\sqrt{2} = m_0/n_0$

Thus, $2(n_0)^2 = (m_0)^2$ Hence $(m_0)^2$ is even.

and $m_0 = 2\,m_1$ for some m_1.

Now $2(n_0)^2 = 4(m_1)^2$ and so

$n_0^2 = 2(m_1)^2$ hence even from

which it follows That n_0 is even.

<u>Recap</u> Whenever $\sqrt{2} = m_0/n_0$ then

it is necessarily The case That

both m_0 and n_0 are even.

This is absurd because Then the fraction for $\sqrt{2}$ would not have an equivalent in lowest terms.

$\square$

$P \iff \neg(\neg P) \iff (\neg P \Rightarrow \underline{\text{false}})$

Proof by contradiction

The strategy for proof by contradiction:

To prove a goal P by contradiction is to prove the equivalent statement $\neg\mathrm{P} \implies \mathbf{false}$

which is proved by assuming $\neg P$ and establishing a contradiction.

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Proof pattern:

In order to prove

$$\mathsf{P}$$

1. Write: We use proof by contradiction. So, suppose P is false.
2. Deduce a logical contradiction.
3. Write: This is a contradiction. Therefore, P must be true.

Scratch work:

Before using the strategy

Assumptions	Goal
	P
$\vdots$	

After using the strategy

Assumptions	Goal
	contradiction
$\vdots$	
$\neg P$	

Theorem 35 *For all statements* P *and* Q,

$$(\neg Q \implies \neg P) \implies (P \implies Q) \ .$$

PROOF: Assume $\boxed{(\neg Q \Rightarrow \neg P)}$ ①. Assume $\boxed{P}$ ②.

Our goal is to show Q. By contradiction assume $\boxed{\neg Q}$ ③.

From ③ and ①, we conclude $\neg P \Leftrightarrow \boxed{(P \Rightarrow \mathit{false})}$ ④

From ② and ④, we conclud false. Hence a contradiction.

□

Every rational number can be expressed as a fraction in lowest terms (†).

Lemma 36 *A positive real number* x *is rational iff*

(*) $\begin{cases} \exists \textit{positive integers } m, n : \\ \quad x = m/n \;\&\; \neg(\exists \textit{prime } p : p \mid m \;\&\; p \mid n) \end{cases}$

PROOF: ($\Leftarrow$) Easy.

($\Rightarrow$) Assume x rational

$\iff \boxed{(\exists k, l .\ x = k/l)}$ (1)

By contradiction, we assume $\neg$(*)

$\iff \forall$ pos. int $m, n .\ \neg\left(x = m/n \;\&\; \neg(\text{———})\right)$

$\iff \forall$ pos int $m, n .\ \neg(x = m/n) \vee \neg\neg(\text{——})$

$\iff \forall$ pos int $m, n .\ \neg(x = m/n) \vee (\text{————})$

② $\iff$ $\forall$ pos. ints m, n.
$$\boxed{\left(x = m/n \Rightarrow \exists \text{ prime } p.\ p \mid m \;\&\; p \mid n\right)}$$

want to show that it leads to contradiction

By ①, let k and l be such that $x = k/l$, k, l pos. ints. Then, by ② specialised

$\exists$ prime p_0. $p_0 \mid k$ & $p_0 \mid l$.

for k and l

$$x = k/l = p_0 \cdot k_0 / p_0 \cdot l_0 = k_0/l_0 \underset{②}{=} p_1 \cdot k_1 / p_1 \cdot l_1 = k_1/l_1$$

$$k = p_0 k_0 = p_0 p_1 k_1 = p_0 p_1 p_2 k_2$$

$$= p_0 \cdot p_1 \cdots p_k \cdot k'$$

$$\geq 2^k \quad \lightning$$

$$\square$$