# Existential quantification

Existential statements are of the form

> **there exists** an individual $x$ in the universe of discourse for which the property $P(x)$ holds

or, in other words,

> **for some** individual $x$ in the universe of discourse, the property $P(x)$ holds

or, in symbols,

$$\boxed{\exists x.\, P(x)}$$

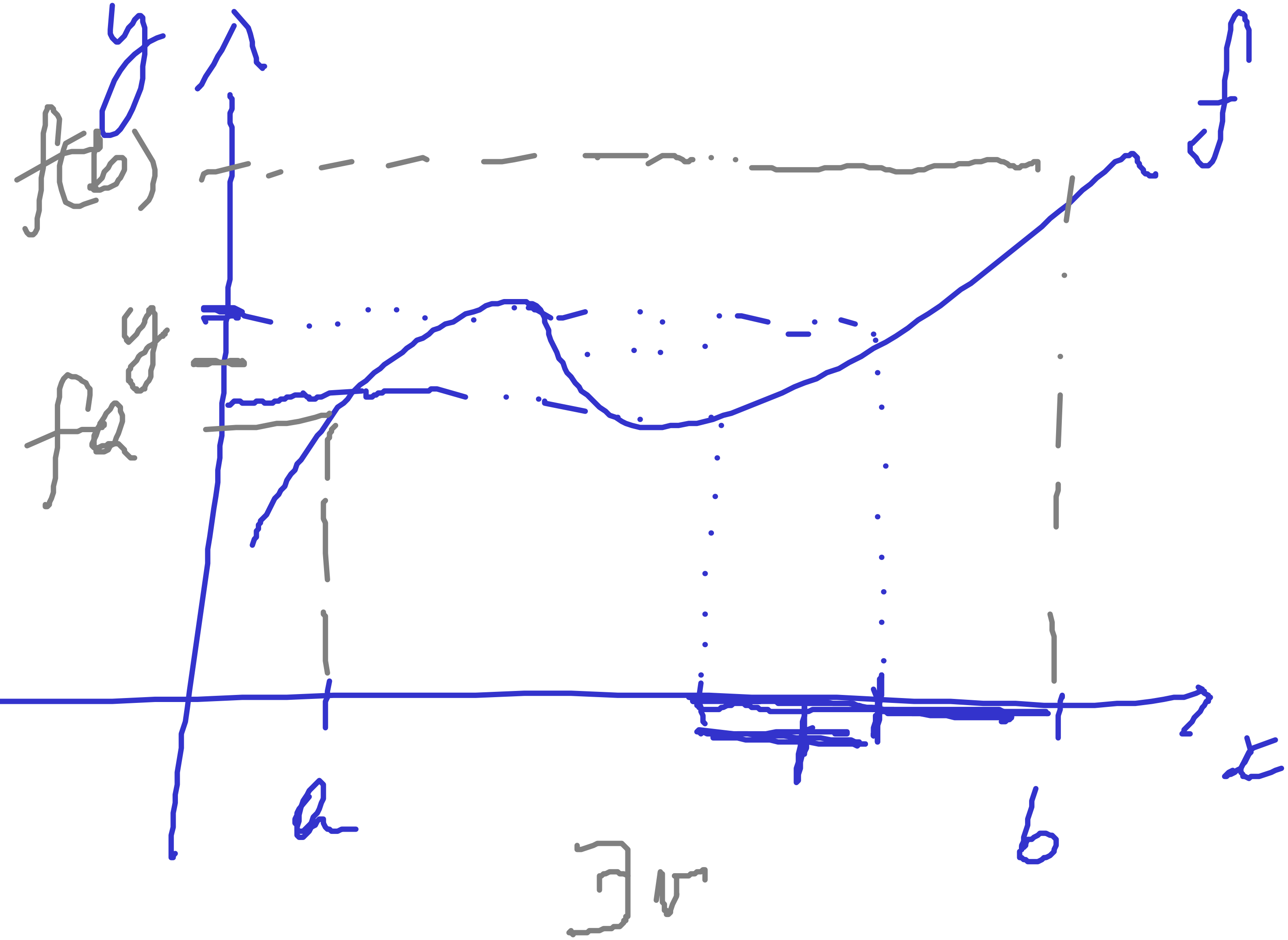
equivalently

$$\exists y.\, P(y)$$

**Theorem 21 (Intermediate value theorem)** *Let* $f$ *be a real-valued continuous function on an interval* $[a, b]$*. For every* $y$ *in between* $f(a)$ *and* $f(b)$*, there exists* $v$ *in between* $a$ *and* $b$ *such that* $f(v) = y$*.*

**Intuition:**

$\forall y$



**The main proof strategy for existential statements:**

To prove a goal of the form

$$\exists x.\, P(x)$$

find a *witness* for the existential statement; that is, a value of $x$, say $w$, for which you think $P(x)$ will be true, and show that indeed $P(w)$, i.e. the predicate $P(x)$ instantiated with the value $w$, holds.

**Proof pattern:**

In order to prove

$$\exists x.\, P(x)$$

1. Write: $\mathrm{Let}\ w = \ldots$ (the witness you decided on).
2. Provide a proof of $P(w)$.

**Scratch work:**

Before using the strategy

| Assumptions | Goal |
| --- | --- |
| | $\exists x.\, P(x)$ |
| $\vdots$ | |

After using the strategy

| Assumptions | Goals |
| --- | --- |
| | $P(w)$ |
| $\vdots$ | |

$w = \ldots$ (the witness you decided on)

**Proposition 22** *For every positive integer $k$, there exist natural numbers $i$ and $j$ such that $4 \cdot k = i^2 - j^2$.*

PROOF OF Proposition 22:

$\forall$ pos. int $k$. $\exists$ nat $i, j$. $4k = i^2 - j^2$

| $k$ | $i$ | $j$ | $i^2 - j^2$ | $4k$ |
|---|---|---|---|---|
| 1 | 2 | 0 | $4-0$ | 4 |
| 2 | 3 | 1 | $9-1$ | 8 |
| 3 | 4 | 2 | $\vdots$ | 12 |
| 4 | 5 | 3 | $\vdots$ | 16 |
| $k$ | $k+1$ | $k-1$ | | |

Let $k$ be an arbitrary positive integer. Let $i = k+1$ and $j = k-1$, we will show that $4k = i^2 - j^2$. Indeed, we calculate

$$i^2 - j^2 = (k+1)^2 - (k-1)^2$$

$$= \ldots \qquad = 4k$$

$\square$

**The use of existential statements:**

To use an assumption of the form $\exists x.\, P(x)$, introduce a new variable $x_0$ into the proof to stand for some individual for which the property $P(x)$ holds. This means that you can now assume $P(x_0)$ true.

$(d \mid a \iff \exists \text{ int } k.\ a = k \cdot d.)$

**Theorem 24** *For all integers* $l$, $m$, $n$, *if* $l \mid m$ *and* $m \mid n$ *then* $l \mid n$.

PROOF: $\forall \text{ int } l, m, n.$

$$\left[ (\exists \text{ int } i.\ m = i \cdot l) \ \& \ (\exists \text{ int } j.\ n = j \cdot m) \right]$$

$$\Longrightarrow (\exists \text{ int } k.\ n = k \cdot l)$$

Let $l, m, n$ be arbitrary integers. Assume

① $\exists \text{ int } i.\ m = i \cdot l$

and

② $\exists \text{ int } j.\ n = j \cdot m$

We prove ⊛ $\exists \text{ int } k.\ n = k \cdot l$

$$\boxed{\begin{aligned} n = j \cdot m &= j \cdot (i \cdot l) \\ &= (j \cdot i) \cdot l \end{aligned}}$$

By (1), let $i_0$ be such that $m = i_0 \cdot l$.

By (2), let $j_0$ be such that $n = j_0 \cdot m$.

Hence, $n = (j_0 \cdot i_0) \cdot l$ and $k = j_0 \cdot i_0$ is a witness for the existential (*)

$\square$

Example: Let n be a positive integer.
If I put n+1 balls into n buckets
Then There is a bucket with more
Than on ball.

- PIGEONHOLE PRINCIPLE

# Disjunction

Disjunctive statements are of the form

$$\boxed{P \text{ or } Q}$$

or, in other words,

$$\boxed{\text{either } P,\ Q, \text{ or both hold}}$$

or, in symbols,

$$\boxed{P \vee Q}$$

**The main proof strategy for disjunction:**

To prove a goal of the form

$$P \vee Q$$

you may

1. try to prove $P$ (if you succeed, then you are done); or
2. try to prove $Q$ (if you succeed, then you are done); otherwise
3. break your proof into cases; proving, in each case, either $P$ or $Q$.

**Proposition 25** *For all integers* $n$, *either* $n^2 \equiv 0 \pmod 4$ *or* $n^2 \equiv 1 \pmod 4$.

$n^2$ is of the form $4i$ for some $i$

PROOF: $n^2$ is of the form $4j+1$ for some $j$

Let $n$ be an arbitrary integer.

$\boxed{?}$ Is $n^2 = 4i$ for some $i$?

| $n$ | $n^2$ | |
|---|---|---|
| 0 | 0 | ✓ |
| 1 | 1 | ✗ $\equiv 1$ |
| 2 | 4 | ✓ |
| 3 | 9 | ✗ $\equiv 1$ |
| 4 | 16 | ✓ |
| 5 | 25 | ✗ $\equiv 1$ |

Consider two cases:

(1) $n$ is even; that is of the form $2i$ for some $i$. Then,

$$n^2 = (2i)^2 = 4 \cdot i^2 \equiv 0 \pmod 4$$

(2) $n$ is odd; That is, of the form $2j+1$ for some $j$. Then

$$n^2 = (2j+1)^2 = 4j^2 + 4j + 1$$
$$= 4(j^2+j) + 1 \equiv 1 \pmod 4$$

$\square$

**The use of disjunction:**

To use a disjunctive assumption

$$P_1 \vee P_2$$

to establish a goal $Q$, consider the following two cases in turn: $(i)$ assume $P_1$ to establish $Q$, and $(ii)$ assume $P_2$ to establish $Q$.

**Scratch work:**

Before using the strategy

| Assumptions | Goal |
|---|---|
| | $Q$ |
| $\vdots$ | |
| $P_1 \vee P_2$ | |

After using the strategy

| Assumptions | Goal | Assumptions | Goal |
|---|---|---|---|
| | $Q$ | | $Q$ |
| $\vdots$ | | $\vdots$ | |
| $P_1$ | | $P_2$ | |

**Proof pattern:**

In order to prove $Q$ from some assumptions amongst which there is

$$P_1 \vee P_2$$

write: We prove the following two cases in turn: ($i$) that assuming $P_1$, we have $Q$; and ($ii$) that assuming $P_2$, we have $Q$. Case ($i$): Assume $P_1$. and provide a proof of $Q$ from it and the other assumptions. Case ($ii$): Assume $P_2$. and provide a proof of $Q$ from it and the other assumptions.