

□ How do we prove something is a lub?

Example

$$f = \text{lub} \bigcup_n (f_n d)$$

is a lub of

$$f_0 \leq f_1 \leq \dots \leq f_n \leq \dots \quad (*)$$

(1) Show  $f$  is an upper bound of  $(*)$ ; That is  $\forall n. f_n \leq f$

$$\text{iff } \forall n \forall d \quad f_n(d) \leq f(d) = \text{lub} \bigcup_n (f_n d)$$

which is the case because  $\forall i. x_i \leq \bigcup_n x_n$ .

(2) Show for all upper bounds  $g$   
(i.e.  $g$  s.t.  $f_n \leq g$   $\forall n$ ) we have  
 $f \leq g$ . iff  $\forall d \quad f(d) \leq g(d)$

$$\text{" } \bigcup_n (f_n d)$$

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$$\forall n \quad f_n(d) \leq g(d)$$

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$$\bigcup_n (f_n d) \leq g(d)$$

## Continuity of the fixpoint operator

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Let  $D$  be a domain.

By Tarski's Fixed Point Theorem we know that each continuous function  $f \in (D \rightarrow D)$  possesses a least fixed point,  $fix(f) \in D$ .

**Proposition.** *The function*

$$fix : (D \rightarrow D) \rightarrow D$$

*is continuous.*

(1)  $\text{fix}$  is monotone.

$$f \sqsubseteq g \Rightarrow \underline{\text{fix}}(f) \sqsubseteq \underline{\text{fix}}(g)$$

$$\bigcup_n f(g^n \perp)$$

$\parallel$

$$f(\bigcup_n g^n \perp)$$

$=$

$$f(\text{fix } g) \sqsubseteq \text{fix}(g)$$

$$\bigcup_n g^n \perp$$

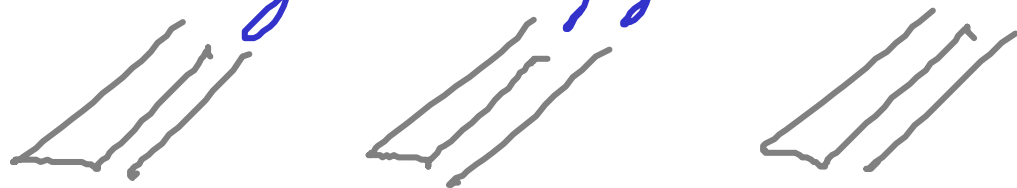
$\parallel$

$$\frac{f(d) \sqsubseteq d}{\underline{\text{fix}}(f) \sqsubseteq d}$$

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$$\underline{\text{fix}}(f) \sqsubseteq \underline{\text{fix}}(g)$$

$$g^n \perp : \quad \perp \leq g \perp \leq g^2 \perp \leq \dots$$



$f \leq g$

$$f(g^n \perp) : \quad f(\perp) \leq f(g \perp) \leq f(g^2 \perp) \leq \dots$$



$$\bigcup_{n \geq 0} f g^n \perp \leq \bigcup_{n \geq 1} g^n \perp = \bigcup_{n \geq 0} g^n \perp$$

(2)  $f$  is preserves lubs.

$$f_0 \leq f_1 \leq \dots \leq f_n \leq \dots$$

$$\underline{f}(\bigsqcup_n f_n) \stackrel{?}{=} \bigsqcup_n (\underline{f} f_n)$$

$$\underline{f}(\underline{f} f) = \underline{f} f$$

Since  $\underline{f}$  is monotone  $\supseteq$  holds. So we

show  $\subseteq \bigsqcup_n \bigsqcup_m f_n(\underline{f} f_m)$

$$\bigsqcup_n f_n(\bigsqcup_m \underline{f}(f_m)) \quad \parallel \quad \bigsqcup_k f_k(\underline{f} f_k)$$

$$\parallel \quad (\bigsqcup_n f_n) (\bigsqcup_n \underline{f}(f_n)) \stackrel{?}{\subseteq} \bigsqcup_n \underline{f} f_n$$

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$$\underline{f}(\bigsqcup_n f_n) \subseteq \bigsqcup_n \underline{f} f_n$$

# ***Topic 4***

## Scott Induction

$$\text{fix}(f) = \bigcup_n f^n(\perp)$$

$$\boxed{\perp \in S} \Rightarrow f(\perp) \in S \Rightarrow \dots \Rightarrow f^n(\perp) \in S$$

## Scott's Fixed Point Induction Principle

Let  $f : D \rightarrow D$  be a continuous function on a domain  $D$ .

For any admissible subset  $S \subseteq D$ , to prove that the least fixed point of  $f$  is in  $S$ , i.e. that

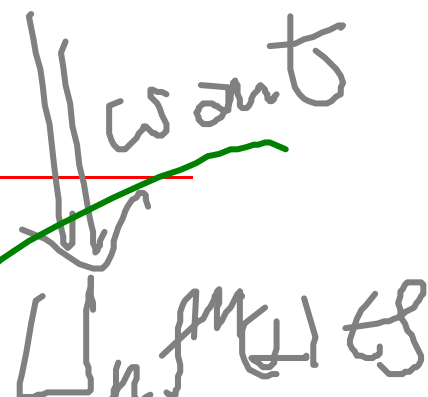
$\text{fix}(f) \in S$ , Guaranteed by  
requiring

it suffices to prove

$$\forall d \in D (d \in S \Rightarrow f(d) \in S)$$

$\bigcup_n d_n \in S$

$$\frac{d \in S \Rightarrow f(d) \in S}{\text{fix}(f) \in S} \quad (S \text{ admissible})$$





## Chain-closed and admissible subsets

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Let  $D$  be a cpo. A subset  $S \subseteq D$  is called **chain-closed** iff for all chains  $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \dots$  in  $D$

$$(\forall n \geq 0 . d_n \in S) \Rightarrow \left( \bigsqcup_{n \geq 0} d_n \right) \in S$$

If  $D$  is a domain,  $S \subseteq D$  is called **admissible** iff it is a chain-closed subset of  $D$  and  $\perp \in S$ .

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A property  $\Phi(d)$  of elements  $d \in D$  is called *chain-closed* (resp. *admissible*) iff  $\{d \in D \mid \Phi(d)\}$  is a *chain-closed* (resp. *admissible*) subset of  $D$ .

## Building chain-closed subsets (I)

Let  $D, E$  be cpos.

Basic relations:

- For every  $d \in D$ , the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{x \in D \mid x \sqsubseteq d\}$$

of  $D$  is chain-closed.

$$\frac{x \sqsubseteq d \implies f(x) \sqsubseteq d}{f \circ \alpha(f) \sqsubseteq d}$$

(1)  $\perp \in \downarrow(d)$  ✓

(2) Let  $d_0 \sqsubseteq d_1 \sqsubseteq \dots \sqsubseteq d_n \sqsubseteq \dots$  in  $\downarrow(d)$   
i.e.  $d_n \sqsubseteq d \ \forall n \implies \bigwedge n d_n \sqsubseteq d$   
 $\implies \bigcup_n d_n \in \downarrow(d)$ .

## Building chain-closed subsets (I)

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Let  $D, E$  be cpos.

### Basic relations:

- For every  $d \in D$ , the subset

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of  $D$  is chain-closed.

- The subsets

$$\{(x, y) \in D \times D \mid x \sqsubseteq y\}$$

and

$$\{(x, y) \in D \times D \mid x = y\}$$

of  $D \times D$  are chain-closed.

The inequality and equality relations are admissible.

$\subseteq \subseteq D \times D$  is chain closed

$$(x_0, y_0) \subseteq (x_1, y_1) \subseteq \dots \subseteq (x_n, y_n) \subseteq \underbrace{\quad}_m \subseteq$$

Show

$$\bigcup_n (x_n, y_n) \text{ in } \subseteq$$

I.e

$$\bigcup_n x_n \subseteq \bigcup_n y_n.$$

i.e

$$\begin{array}{ccc} x_0 & \subseteq & y_0 \\ \uparrow & & \uparrow \\ n_1 & & n_1 \\ x_1 & \subseteq & y_1 \\ \uparrow & & \uparrow \\ n_1 & & n_1 \\ & \vdots & \\ x_n & \subseteq & y_n \\ \uparrow & & \uparrow \\ n_1 & & n_1 \\ & \vdots & \end{array}$$

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$$\bigcup_n x_n \subseteq \bigcup_n y_n$$

## Example (I): Least pre-fixed point property

Let  $D$  be a domain and let  $f : D \rightarrow D$  be a continuous function.

$$\forall d \in D. f(d) \sqsubseteq d \implies \text{fix}(f) \sqsubseteq d$$

Assume  $f(d) \sqsubseteq d$

$$\frac{[x \sqsubseteq d]}{fx \sqsubseteq fd}$$

Scott induction



least pre fixed point property.

$$fx \sqsubseteq d$$

$$x \sqsubseteq d \implies fx \sqsubseteq d$$

$$\text{fix}(f) \sqsubseteq d$$

$\downarrow (d)$  adm.

## Example (I): Least pre-fixed point property

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Let  $D$  be a domain and let  $f : D \rightarrow D$  be a continuous function.

$$\forall d \in D. f(d) \sqsubseteq d \implies \text{fix}(f) \sqsubseteq d$$

Proof by Scott induction.

Let  $d \in D$  be a pre-fixed point of  $f$ . Then,

$$\begin{aligned} x \in \downarrow(d) &\implies x \sqsubseteq d \\ &\implies f(x) \sqsubseteq f(d) \\ &\implies f(x) \sqsubseteq d \\ &\implies f(x) \in \downarrow(d) \end{aligned}$$

Hence,

$$\text{fix}(f) \in \downarrow(d) .$$

$d_0 \subseteq d_1 \subseteq \dots \subseteq d_n \subseteq \dots$  in  $f^{-1}(S)$

$\Rightarrow f d_0 \subseteq f d_1 \subseteq \dots \subseteq f d_n \subseteq \dots$  in  $S$

**Building chain-closed subsets (II)**  $\Rightarrow \bigcup_n (f d_n) \in S$

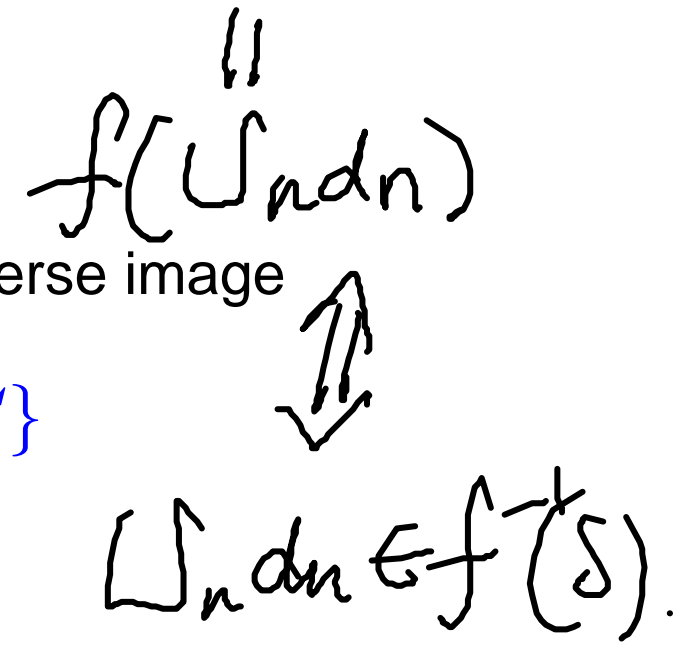
**Inverse image:**

Let  $f : D \rightarrow E$  be a continuous function.

If  $S$  is a chain-closed subset of  $E$  then the inverse image

$$f^{-1}S = \{x \in D \mid f(x) \in S\}$$

is an chain-closed subset of  $D$ .



## Example (II)

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Let  $D$  be a domain and let  $f, g : D \rightarrow D$  be continuous functions such that  $f \circ g \sqsubseteq g \circ f$ . Then,

$$f(\perp) \sqsubseteq g(\perp) \implies \text{fix}(f) \sqsubseteq \text{fix}(g) .$$

Exercise



## Example (II)

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Let  $D$  be a domain and let  $f, g : D \rightarrow D$  be continuous functions such that  $f \circ g \sqsubseteq g \circ f$ . Then,

$$f(\perp) \sqsubseteq g(\perp) \implies \text{fix}(f) \sqsubseteq \text{fix}(g) .$$

Proof by Scott induction.

Consider the admissible property  $\Phi(x) \equiv (f(x) \sqsubseteq g(x))$  of  $D$ .

Since

$$f(x) \sqsubseteq g(x) \implies g(f(x)) \sqsubseteq g(g(x)) \implies f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\text{fix}(g)) \sqsubseteq g(\text{fix}(g)) .$$

## Building chain-closed subsets (III)

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### Logical operations:

- If  $S, T \subseteq D$  are chain-closed subsets of  $D$  then

$$S \cup T \quad \text{and} \quad S \cap T$$

are chain-closed subsets of  $D$ .

- If  $\{S_i\}_{i \in I}$  is a family of chain-closed subsets of  $D$  indexed by a set  $I$ , then  $\bigcap_{i \in I} S_i$  is a chain-closed subset of  $D$ .
- If a property  $P(x, y)$  determines a chain-closed subset of  $D \times E$ , then the property  $\forall x \in D. P(x, y)$  determines a chain-closed subset of  $E$ .

$\text{Pr}(\text{fix } \omega, \llbracket Y := X * Y; X := X - 1 \rrbracket) \in S$

### Example (III): Partial correctness

Let  $\mathcal{F} : \text{State} \rightarrow \text{State}$  be the denotation of

$\mathcal{F} = \llbracket \text{while } X > 0 \text{ do } (Y := X * Y; X := X - 1) \rrbracket$

For all  $x, y \geq 0$ ,

$\mathcal{F}[X \mapsto x, Y \mapsto y] \downarrow$

$\implies \mathcal{F}[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y]$

$S = \{ \omega \mid \omega(x, y) \downarrow \implies \omega(x, y) = (0, !x \cdot y) \}$

Recall that

$$\mathcal{F} = \text{fix}(f)$$

where  $f : (\text{State} \rightarrow \text{State}) \rightarrow (\text{State} \rightarrow \text{State})$  is given by

$$f(w) = \lambda(x, y) \in \text{State}. \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$

Proof by Scott induction.

We consider the admissible subset of  $(State \rightarrow State)$  given by

$$S = \left\{ w \mid \begin{array}{l} \forall x, y \geq 0. \\ w[X \mapsto x, Y \mapsto y] \downarrow \\ \Rightarrow w[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y] \end{array} \right\}$$

and show that

$$w \in S \implies f(w) \in S .$$