$\boxed{?}$ How do we prove something is a lub?

Example

$$f = \lambda d.\ \bigsqcup_n (f_n\, d)$$

is a lub of

$$f_0 \sqsubseteq f_1 \sqsubseteq \ldots \sqsubseteq f_n \sqsubseteq \ldots \quad (*)$$

(1) Show f is an upper bound of $(*)$; that is $\forall n.\ f_n \sqsubseteq f$

iff $\forall n\ \forall d\quad f_n(d) \sqsubseteq f(d) = \bigsqcup_n (f_n\, d)$

which is the case because $\forall i\ \ x_i \sqsubseteq \bigsqcup_n x_n$.

(2) Show for all upper bounds g (i.e. g s.t. $f_n \sqsubseteq g \ \forall n$) we have $f \sqsubseteq g$ iff $\forall d \ f(d) \sqsubseteq g(d)$

$$f(d) = \bigsqcup_n (f_n d)$$

$$\frac{\forall n \quad f_n(d) \sqsubseteq g(d)}{\bigsqcup_n (f_n d) \sqsubseteq g(d)}$$

Continuity of the fixpoint operator

Let D be a domain.

By Tarski's Fixed Point Theorem we know that each continuous function $f \in (D \to D)$ possesses a least fixed point, $\mathit{fix}(f) \in D$.

Proposition. *The function*

$$\mathit{fix} : (D \to D) \to D$$

is continuous.

(1) fix is monotone.

$$f \sqsubseteq g \Rightarrow \underline{fix}(f) \sqsubseteq \underline{fix}(g)$$

$$\bigsqcup_n f(g^n \bot)$$

$$\shortparallel$$

$$f(\bigsqcup_n g^n \bot)$$

$$=$$

$$\bigsqcup_n g^n(\bot)$$

$$\shortparallel$$

$$\frac{f(\mathit{fix}\, g) \sqsubseteq \mathit{fix}(g)}{\mathit{fix}(f) \sqsubseteq \mathit{fix}(g)}$$

$$\frac{f(d) \sqsubseteq d}{\mathit{fix}(f) \sqsubseteq d}$$

$g^n(\perp)$: $\perp \sqsubseteq g\perp \sqsubseteq gg\perp \sqsubseteq \cdots$

$f \sqsubseteq g$

$f(g^n \perp)$: $f(\perp) \sqsubseteq f(g\perp) \sqsubseteq f(gg\perp) \sqsubseteq \cdots$

$$\bigsqcup_{n\geq 0} fg^n\perp \sqsubseteq \bigsqcup_{n\geq 1} g^n\perp = \bigsqcup_{n\geq 0} g^n\perp$$

(2) fix is preserves lubs.

$f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots$

$$f(\underline{fix}\, f) = \underline{fix}\, f$$

$$\underline{fix}\left(\bigsqcup_n f_n\right) \overset{?}{=} \bigsqcup_n (\underline{fix}\, f_n)$$

Since $\underline{fix}$ is monotone $\sqsupseteq$ holds. So we show $\sqsubseteq$

$$\bigsqcup_n \bigsqcup_m f_n(\underline{fix}\, f_m) = \bigsqcup_k f_k(\underline{fix}\, f_k)$$

$$\bigsqcup_n f_n\left(\bigsqcup_m \underline{fix}(f_m)\right) = \bigsqcup_n \bigsqcup_m f_n(\underline{fix}\, f_m)$$

$$\bigsqcup_n f_n\left(\bigsqcup_m \underline{fix}(f_m)\right) = \left(\bigsqcup_n f_n\right)\left(\bigsqcup_n \underline{fix}(f_n)\right)$$

$$\bigsqcup_k f_k(\underline{fix}\, f_k) = \bigsqcup_n \underline{fix}\, f_n$$

$$\frac{\left(\bigsqcup_n f_n\right)\left(\bigsqcup_n \underline{fix}(f_n)\right) \overset{?}{\sqsubseteq} \bigsqcup_n \underline{fix}\, f_n}{\underline{fix}\left(\bigsqcup_n f_n\right) \sqsubseteq \bigsqcup_n \underline{fix}(f_n)}$$

Topic 4

Scott Induction

$fix(f) = \bigsqcup_n f^n(\perp)$ $\boxed{\perp \in S} \Rightarrow f(\perp) \in S \Rightarrow \dots \Rightarrow f^n(\perp) \in S$

Scott's Fixed Point Induction Principle

want $\bigsqcup_n f^n(\perp) \in S$

Let $f : D \to D$ be a continuous function on a domain D.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of f is in S, *i.e.* that

$$fix(f) \in S \,,$$

Guaranteed by requiring $\boxed{\dfrac{d_0 \sqsubseteq \dots \sqsubseteq d_n \sqsubseteq \dots \text{ in } S}{\bigsqcup_n d_n \text{ in } S}}$

it suffices to prove

$$\forall d \in D \; (d \in S \;\Rightarrow\; f(d) \in S) \,.$$

$$\frac{d \in S \Rightarrow f(d) \in S}{fix(f) \in S} \quad (S \text{ admissible})$$

Chain-closed and admissible subsets

Let D be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D

$$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \Big(\bigsqcup_{n \geq 0} d_n\Big) \in S$$

If D is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of D and $\perp \in S$.

A property $\Phi(d)$ of elements $d \in D$ is called *chain-closed* (resp. *admissible*) iff $\{d \in D \mid \Phi(d)\}$ is a *chain-closed* (resp. *admissible*) subset of D.

Building chain-closed subsets (I)

Let D, E be cpos.

Basic relations:

- For every $d \in D$, the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{ x \in D \mid x \sqsubseteq d \}$$

of D is chain-closed.

$$\boxed{\frac{x \sqsubseteq d \Rightarrow f(x) \sqsubseteq d}{fix(f) \sqsubseteq d}}$$

(1) $\perp \in \downarrow(d)$ ✓

(2) Let $d_0 \sqsubseteq d_1 \sqsubseteq \dots \sqsubseteq d_n \sqsubseteq \dots$ in $\downarrow(d)$

i.e. $d_n \sqsubseteq d \;\; \forall n \Rightarrow \bigsqcup_n d_n \sqsubseteq d$

$\Rightarrow \bigsqcup_n d_n \in \downarrow(d)$.

Building chain-closed subsets (I)

Let D, E be cpos.

Basic relations:

- For every $d \in D$, the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{\, x \in D \mid x \sqsubseteq d \,\}$$

of D is chain-closed.

- The subsets

$$\{(x, y) \in D \times D \mid x \sqsubseteq y\}$$

and

$$\{(x, y) \in D \times D \mid x = y\}$$

of $D \times D$ are chain-closed.

The inequality and equality relations are admissible.

$\sqsubseteq \; \subseteq D \times D$ is chain closed

$(x_0, y_0) \sqsubseteq (x_1, y_1) \sqsubseteq \cdots \sqsubseteq (x_n, y_n) \sqsubseteq \underbrace{\;\; \text{in} \; \sqsubseteq}$

Show

$\bigsqcup_n (x_n, y_n)$ in $\sqsubseteq$

I.e $\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n$.

i.e

$$
\begin{array}{ccc}
x_0 & \sqsubseteq & y_0 \\
\rotatebox{-90}{$\sqsubseteq$} & & \rotatebox{-90}{$\sqsubseteq$} \\
x_1 & \sqsubseteq & y_1 \\
\rotatebox{-90}{$\sqsubseteq$} & \vdots & \rotatebox{-90}{$\sqsubseteq$} \\
x_n & \sqsubseteq & y_n \\
\rotatebox{-90}{$\sqsubseteq$} & \vdots & \rotatebox{-90}{$\sqsubseteq$} \\
\hline
\multicolumn{3}{c}{\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n}
\end{array}
$$

Example (I): Least pre-fixed point property

Let D be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\, f(d) \sqsubseteq d \implies \mathit{fix}(f) \sqsubseteq d$$

Assume $f(d) \sqsubseteq d$

$$\dfrac{\dfrac{\dfrac{\dfrac{[x \sqsubseteq d]}{fx \sqsubseteq fd} \qquad f(d) \sqsubseteq d}{f(x) \sqsubseteq d}}{x \sqsubseteq d \Rightarrow f(x) \sqsubseteq d}}{\mathit{fix}(f) \sqsubseteq d}\ \downarrow(d) \text{ adm.}$$

Scott induction

$\Downarrow$

least pre-fixed point property.

Example (I): Least pre-fixed point property

Let D be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\, f(d) \sqsubseteq d \implies \mathit{fix}(f) \sqsubseteq d$$

Proof by Scott induction.

Let $d \in D$ be a pre-fixed point of f. Then,

$$\begin{aligned} x \in {\downarrow}(d) \quad & \implies \quad x \sqsubseteq d \\ & \implies \quad f(x) \sqsubseteq f(d) \\ & \implies \quad f(x) \sqsubseteq d \\ & \implies \quad f(x) \in {\downarrow}(d) \end{aligned}$$

Hence,

$$\mathit{fix}(f) \in {\downarrow}(d) \ .$$

$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq$ in $f^{-1}(S)$

$\Rightarrow f d_0 \sqsubseteq f d_1 \sqsubseteq \cdots \sqsubseteq f d_n \sqsubseteq \cdots$ in S

Building chain-closed subsets (II) $\Rightarrow \bigsqcup_n (f d_n) \in S$

$$\parallel$$

$$f(\bigsqcup_n d_n)$$

Inverse image:

Let $f : D \to E$ be a continuous function.

If S is a chain-closed subset of E then the inverse image

$$f^{-1}S \;=\; \{x \in D \mid f(x) \in S\}$$

$$\Updownarrow$$

is an chain-closed subset of D.

$$\bigsqcup_n d_n \in f^{-1}(S).$$

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

Exercise

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

Proof by Scott induction.

Consider the admissible property $\Phi(x) \equiv \big(f(x) \sqsubseteq g(x)\big)$ of D.

Since

$$f(x) \sqsubseteq g(x) \Rightarrow g(f(x)) \sqsubseteq g(g(x)) \Rightarrow f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\mathit{fix}(g)) \sqsubseteq g(\mathit{fix}(g)) \ .$$

Building chain-closed subsets (III)

Logical operations:

- If $S, T \subseteq D$ are chain-closed subsets of D then

  $$S \cup T \qquad \text{and} \qquad S \cap T$$

  are chain-closed subsets of D.

- If $\{ S_i \}_{i \in I}$ is a family of chain-closed subsets of D indexed by a set I, then $\bigcap_{i \in I} S_i$ is a chain-closed subset of D.

- If a property $P(x, y)$ determines a chain-closed subset of $D \times E$, then the property $\forall x \in D.\ P(x, y)$ determines a chain-closed subset of E.

$$\mathrm{fix}\left(f_{[\![X>0]\!]}, [\![Y := X*Y; X := X-1]\!]\right) \in S$$

Example (III): Partial correctness

Let $\mathcal{F} : State \rightharpoonup State$ be the denotation of

$$\mathcal{F} = [\![\textbf{while } X > 0 \textbf{ do } (Y := X * Y; X := X - 1)]\!]$$

For all $x, y \geq 0$,

$$\mathcal{F}[X \mapsto x, Y \mapsto y] \downarrow$$

$$\implies \mathcal{F}[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y].$$

$$S = \{ w \mid w(x,y)\downarrow \implies w(x,y) = (0, !x \cdot y) \}$$

Recall that

$$\mathcal{F} = \mathit{fix}(f)$$

where $f : (State \rightharpoonup State) \rightarrow (State \rightharpoonup State)$ is given by

$$f(w) = \lambda(x,y) \in State. \begin{cases} (x,y) & \text{if } x \leq 0 \\ w(x-1, x \cdot y) & \text{if } x > 0 \end{cases}$$

Proof by Scott induction.

We consider the admissible subset of $(State \rightharpoonup State)$ given by

$$S = \left\{ w \;\middle|\; \begin{array}{l} \forall x, y \geq 0. \\ \quad w[X \mapsto x, Y \mapsto y]\downarrow \\ \qquad \Rightarrow w[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y] \end{array} \right\}$$

and show that

$$w \in S \implies f(w) \in S \ .$$