# Complexity Theory

# Lecture 1

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http://www.cl.cam.ac.uk/teaching/1314/Complexity/

# Texts

The main texts for the course are:

*Computational Complexity*.
Christos H. Papadimitriou.

*Introduction to the Theory of Computation*.
Michael Sipser.

Other useful references include:

*Computers and Intractability: A guide to the theory of NP-completeness*.
Michael R. Garey and David S. Johnson.

*Computational complexity: a conceptual perspective*.
O. Goldreich.

*Computability and Complexity from a Programming Perspective*.
Neil Jones.

# Outline

A rough lecture-by-lecture guide, with relevant sections from the text by Papadimitriou (or Sipser, where marked with an S).

- **Algorithms and problems.** 1.1–1.3.
- **Time and space.** 2.1–2.5, 2.7.
- **Time Complexity classes.** 7.1, S7.2.
- **Nondeterminism.** 2.7, 9.1, S7.3.
- **NP-completeness.** 8.1–8.2, 9.2.
- **Graph-theoretic problems.** 9.3

# Outline - contd.

- **Sets, numbers and scheduling.** 9.4
- **coNP.** 10.1–10.2.
- **Cryptographic complexity.** 12.1–12.2.
- **Space Complexity** 7.1, 7.3, S8.1.
- **Hierarchy** 7.2, S9.1.
- **Descriptive Complexity** 5.7, 8.3.

# Algorithms and Problems

> Insertion Sort runs in time $O(n^2)$, while Merge Sort is an $O(n \log n)$ algorithm.

The first half of this statement is short for:

> If we count the number of steps performed by the Insertion Sort algorithm on an input of size $n$, taking the largest such number, from among all inputs of that size, then the function of $n$ so defined is eventually bounded by a constant multiple of $n^2$.

It makes sense to compare the two algorithms, because they seek to solve the same problem.

But, what is the complexity of the sorting problem?

# Lower and Upper Bounds

What is the running time complexity of the fastest algorithm that sorts a list?

By the analysis of the Merge Sort algorithm, we know that this is no worse than $O(n \log n)$.

The complexity of a particular algorithm establishes an *upper bound* on the complexity of the problem.

To establish a *lower bound*, we need to show that no possible algorithm, including those as yet undreamed of, can do better.

In the case of sorting, we can establish a lower bound of $\Omega(n \log n)$, showing that Merge Sort is asymptotically optimal.

Sorting is a rare example where known upper and lower bounds match.

# Review

The complexity of an algorithm (whether measuring number of steps, or amount of memory) is usually described asymptotically:

**Definition**

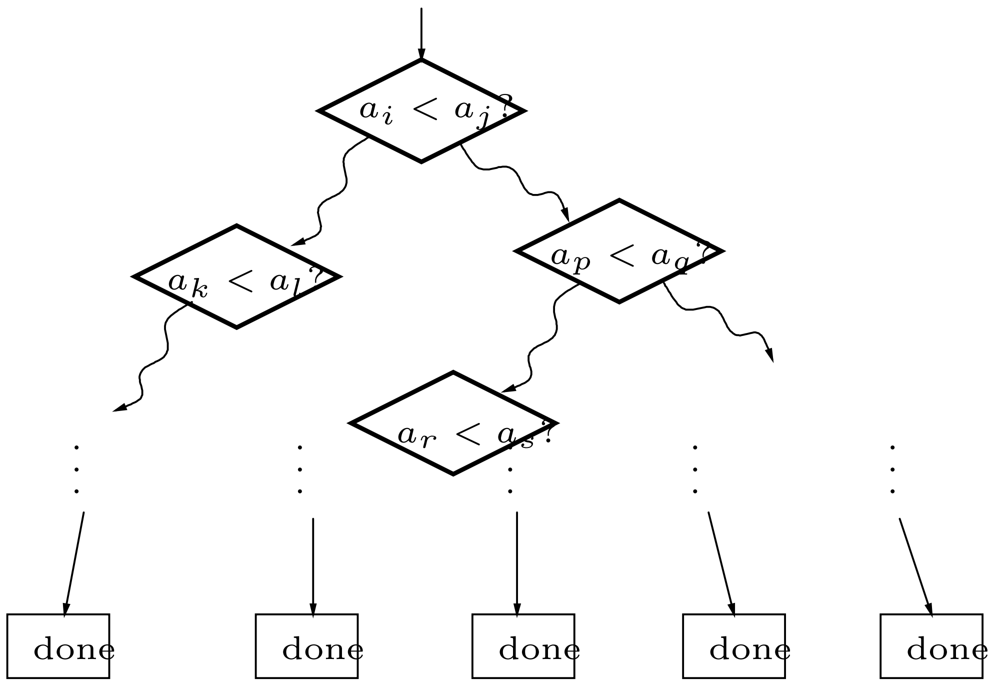
For functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$, we say that:

- $f = O(g)$, if there is an $n_0 \in \mathbb{N}$ and a constant $c$ such that for all $n > n_0$, $f(n) \leq cg(n)$;
- $f = \Omega(g)$, if there is an $n_0 \in \mathbb{N}$ and a constant $c$ such that for all $n > n_0$, $f(n) \geq cg(n)$.
- $f = \theta(g)$ if $f = O(g)$ and $f = \Omega(g)$.

Usually, $O$ is used for upper bounds and $\Omega$ for lower bounds.

# Lower Bound on Sorting

An algorithm $A$ sorting a list of $n$ distinct numbers $a_1, \ldots, a_n$.




To work for all permutations of the input list, the tree must have at least $n!$ leaves and therefore height at least $\log_2(n!) = \theta(n \log n)$.

# Travelling Salesman

Given

- $V$ — a set of nodes.
- $c : V \times V \to \mathbb{N}$ — a cost matrix.

Find an ordering $v_1, \ldots, v_n$ of $V$ for which the total cost:

$$c(v_n, v_1) + \sum_{i=1}^{n-1} c(v_i, v_{i+1})$$

is the smallest possible.

# Complexity of TSP

Obvious algorithm: Try all possible orderings of $V$ and find the one with lowest cost.
The worst case running time is $\theta(n!)$.

Lower bound: An analysis like that for sorting shows a lower bound of $\Omega(n \log n)$.

Upper bound: The currently fastest known algorithm has a running time of $O(n^2 2^n)$.

Between these two is the chasm of our ignorance.

# Formalising Algorithms

To prove a lower bound on the complexity of a problem, rather than a specific algorithm, we need to prove a statement about all algorithms for solving it.

In order to prove facts about all algorithms, we need a mathematically precise definition of algorithm.

We will use the *Turing machine*.

> The simplicity of the Turing machine means it's not useful for actually expressing algorithms, but very well suited for proofs about all algorithms.

# Turing Machines

For our purposes, a Turing Machine consists of:

- $Q$ — a finite set of states;
- $\Sigma$ — a finite set of symbols, including $\sqcup$.
- $s \in Q$ — an initial state;
- $\delta : (Q \times \Sigma) \rightarrow (Q \cup \{a, r\}) \times \Sigma \times \{L, R, S\}$

  A transition function that specifies, for each state and symbol a next state (or accept acc or reject rej), a symbol to overwrite the current symbol, and a direction for the tape head to move ($L$ – left, $R$ – right, or $S$ - stationary)

# Configurations

A complete description of the configuration of a machine can be given if we know what state it is in, what are the contents of its tape, and what is the position of its head. This can be summed up in a simple triple:

**Definition**

A *configuration* is a triple $(q, w, u)$, where $q \in Q$ and $w, u \in \Sigma^\star$

The intuition is that $(q, w, u)$ represents a machine in state $q$ with the string $wu$ on its tape, and the head pointing at the last symbol in $w$.

The configuration of a machine completely determines the future behaviour of the machine.