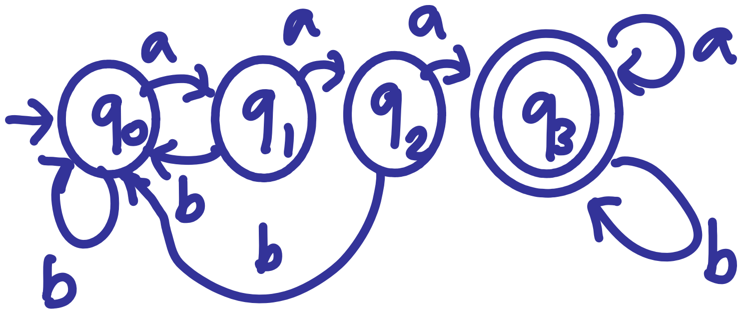
Previously on RLFA...

REGULAR EXPRESSIONS r (eg. $(a|b)^*aaa(a|b)^*$)

FINITE AUTOMATA M (eg.



)

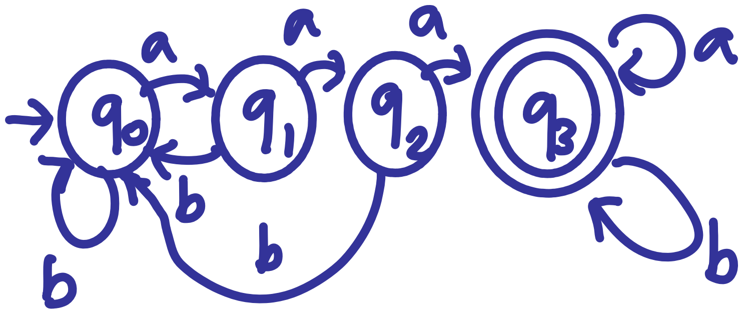
REGULAR LANGUAGE = set of strings of the form $L(M)$ (all strings accepted by M) for some f.a.

REGULAR EXPRESSIONS r

(eg. $(a|b)^*aaa(a|b)^*$)

FINITE AUTOMATA M

(eg.



)

KLEENE :

(a) For all r, can construct M with $L(M) = L(r)$

(b) For all M, can construct r with $L(r) = L(M)$

Typical application: lexical analysis at start of compilation

- PL definition specifies legal tokens (keywords, identifiers, etc.) using a reg. exp.
- Lexical analyser splits a character stream into a stream of tokens by constructing a f.a. from the reg. exp.

Examples of non-regular languages

- The set of strings over $\{(,), a, b, \ldots, z\}$ in which the parentheses '$($' and '$)$' occur well-nested.
- The set of strings over $\{a, b, \ldots, z\}$ which are palindromes, i.e. which read the same backwards as forwards.
- $\{a^n b^n \mid n \geq 0\}$

The Pumping Lemma

For every regular language L, there is a number $\ell \geq 1$ satisfying the ***pumping lemma property***:

all $w \in L$ with $length(w) \geq \ell$ can be expressed as a concatenation of three strings, $w = u_1 v u_2$, where u_1, v and u_2 satisfy:

- $length(v) \geq 1$
  (i.e. $v \neq \varepsilon$)
- $length(u_1 v) \leq \ell$
- for all $n \geq 0$, $u_1 v^n u_2 \in L$
  (i.e. $u_1 u_2 \in L$, $u_1 v u_2 \in L$ [but we knew that anyway], $u_1 v v u_2 \in L$, $u_1 v v v u_2 \in L$, etc).

Suppose L is L(M) for a DFA M.

If $n \geq \ell =$ number of states of M, then in

$$s_M = \underbrace{q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \cdots \xrightarrow{a_\ell} q_\ell}_{\ell+1 \text{ states}} \cdots \xrightarrow{a_n} q_n \in Accept_M$$

$q_0, \ldots, q_\ell$ can't all be distinct states. So $q_i = q_j$ for some $0 \leq i < j \leq \ell$. So the above transition sequence looks like

$$\begin{tikzcd} s_M = q_0 \arrow[r, "u_1"] & {}^{*}\ q_i \arrow[r, phantom, "="] \arrow[r, bend left=70, "v"] & q_j \arrow[r, "u_2"] & {}^{*}\ q_n \in Accept_M \end{tikzcd}$$

where

$$u_1 \overset{\text{def}}{=} a_1 \ldots a_i \qquad v \overset{\text{def}}{=} a_{i+1} \ldots a_j \qquad u_2 \overset{\text{def}}{=} a_{j+1} \ldots a_n.$$

How to use the Pumping Lemma to prove that a language L is *not* regular

For each $\ell \geq 1$, find some $w \in L$ of length $\geq \ell$ so that

$$(\dagger) \quad \left\{ \begin{array}{l} \text{no matter how } w \text{ is split into three, } w = u_1 v u_2, \\ \text{with } length(u_1 v) \leq \ell \text{ and } length(v) \geq 1, \\ \\ \text{there is some } n \geq 0 \text{ for which } u_1 v^n u_2 \text{ is \textit{not} in } L. \end{array} \right.$$

Examples

(i) $L_1 \overset{\text{def}}{=} \{a^n b^n \mid n \geq 0\}$ *is not regular.*

[For each $\ell \geq 1$, $a^\ell b^\ell \in L_1$ is of length $\geq \ell$ and has property (†) on Slide 31.]

(ii) $L_2 \overset{\text{def}}{=} \{w \in \{a,b\}^* \mid w \textit{ a palindrome}\}$ *is not regular.*

[For each $\ell \geq 1$, $a^\ell b a^\ell \in L_1$ is of length $\geq \ell$ and has property (†).]

(iii) $L_3 \overset{\text{def}}{=} \{a^p \mid p \textit{ prime}\}$ *is not regular.*

[For each $\ell \geq 1$, we can find a prime p with $p > 2\ell$ and then $a^p \in L_3$ has length $\geq \ell$ and has property (†).]

Example of a non-regular language that satisfies the 'pumping lemma property'

$$L \overset{\text{def}}{=} \{c^m a^n b^n \mid m \geq 1 \text{ and } n \geq 0\}$$

$$\cup$$

$$\{a^m b^n \mid m, n \geq 0\}$$

satisfies the pumping lemma property on Slide 29 with $\ell = 1$.

[For any $w \in L$ of length ≥ 1, can take $u_1 = \varepsilon$, $v =$ first letter of w, $u_2 =$ rest of w.]

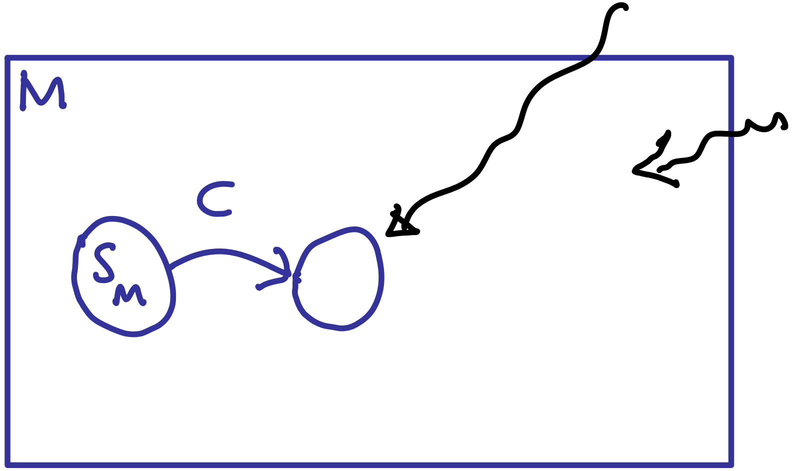
But L is not regular.

↑ Assume $L = L(M)$ for some DFA M
and obtain a contradiction...

Given M, define a new NFA N as follows:

$\text{States}_N = \text{States}_M$

$\text{Accept}_N = \text{Accept}_M$



N starts here

N has only the a & b transitions of M

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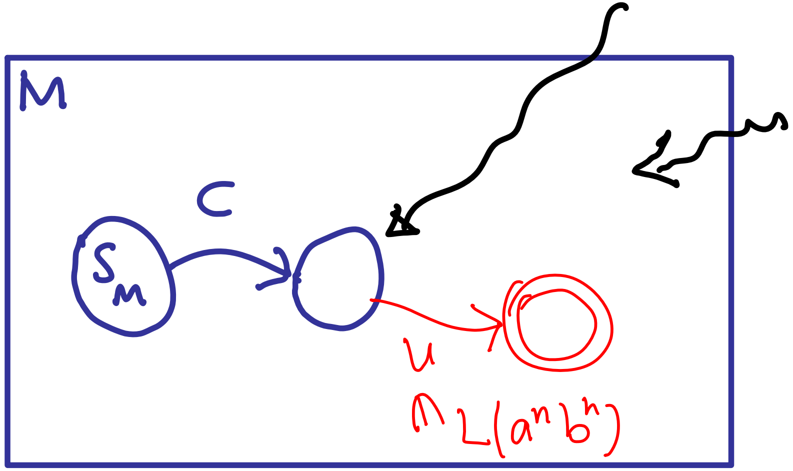

N has only the a & b transitions of M

So $L(N) = \{a^n b^n \mid n \geq 0\}$, contradiction to Pumping Lemma.