Lemma Given an NFA $M$, for each subset $Q \subseteq States_M$ and each pair of states $q, q' \in States_M$, there is a regular expression $r^Q_{q, q'}$ satisfying

$$L(r^Q_{q, q'}) = \{ u \in (\Sigma_M)^* \mid q \xrightarrow{u}^* q' \text{ in } M \text{ with all intermediate states of the sequence in } Q \}.$$

Hence $L(M) = L(r)$, where $r = r_1 \mid \cdots \mid r_k$ and $k = \text{number of accepting states},$

$r_i = r^Q_{s, q_i}$ with $Q = States_M,$

$s = \text{start state},$

$q_i = i\text{th accepting state}.$

(In case $k = 0$, take $r$ to be the regular expression $\emptyset.$)
Direct inspection yields:

\[
\begin{array}{c|ccc}
 r_{i,j}^{\{0\}} & 0 & 1 & 2 \\
\hline
 0 & & & \\
 1 & \emptyset & \varepsilon & a \\
 2 & aa^* & a^*b & \varepsilon \\
\end{array}
\]

\[
\begin{array}{c|ccc}
 r_{i,j}^{\{0,2\}} & 0 & 1 & 2 \\
\hline
 0 & & a^* & a^*b \\
 1 & & & \\
 2 & & & \\
\end{array}
\]
Not(M)

- \(\text{States}_{\text{Not}(M)} \overset{\text{def}}{=} \text{States}_M\)
- \(\Sigma_{\text{Not}(M)} \overset{\text{def}}{=} \Sigma_M\)
- transitions of Not(M) = transitions of M
- start state of Not(M) = start state of M
- \(\text{Accept}_{\text{Not}(M)} = \{q \in \text{States}_M \mid q \notin \text{Accept}_M\}\).

Provided M is a deterministic finite automaton, then \(u\) is accepted by Not(M) iff it is not accepted by M:

\[L(\text{Not}(M)) = \{ u \in \Sigma^* \mid u \notin L(M) \}.\]
\begin{itemize}
\item states of $\text{And}(M_1, M_2)$ are all ordered pairs $(q_1, q_2)$ with $q_1 \in \text{States}_{M_1}$ and $q_2 \in \text{States}_{M_2}$
\item alphabet of $\text{And}(M_1, M_2)$ is the common alphabet of $M_1$ and $M_2$
\item $(q_1, q_2) \xrightarrow{a} (q'_1, q'_2)$ in $\text{And}(M_1, M_2)$ iff $q_1 \xrightarrow{a} q'_1$ in $M_1$ and $q_2 \xrightarrow{a} q'_2$ in $M_2$
\item start state of $\text{And}(M_1, M_2)$ is $(s_{M_1}, s_{M_2})$
\item $(q_1, q_2)$ accepting in $\text{And}(M_1, M_2)$ iff $q_1$ accepting in $M_1$ and $q_2$ accepting in $M_2$.
\end{itemize}