M.Phil. Advanced Computer Science
Michaelmas 2012
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Introductory logic — Exercise sheet 1

Basic set theory and propositional logic

Feel free to write me an e-mail if you have questions about, or corrections to, any of the exercises on this sheet. To indicate the difficulty of the problems, I have marked the (hopefully) most accessible exercises with '-' and the difficult ones (which are optional, if you like) with '+'. The exercises that are unmarked fall somewhere in between.

(-) 1. Show that any recursively enumerable set is countable.

(-) 2. Which of the following propositional formulas are tautologies?

(i) $((p \Rightarrow (q \Rightarrow r)) \Rightarrow (q \Rightarrow (p \Rightarrow r)))$;
(ii) $(((p \Rightarrow q) \Rightarrow r) \Rightarrow ((q \Rightarrow p) \Rightarrow r))$;
(iii) $(((p \Rightarrow q) \Rightarrow p) \Rightarrow p)$;
(iv) $((p \Rightarrow (p \Rightarrow q)) \Rightarrow p)$.

3. Use the Deduction Theorem to show that the formula $(s \Rightarrow \neg\neg s)$ (that is, the converse of the third axiom) is a theorem of propositional logic.

(-) 4. Use the Deduction Theorem to establish the following proof principles for propositional logic:

(i) *Proof by contradiction*: $H \cup \{t\} \vdash \bot$ iff $H \vdash \neg t$.
(ii) *Proof by contrapositive*: $H \cup \{t\} \vdash \neg s$ iff $H \cup \{s\} \vdash \neg t$.

5. Show that $(\bot \Rightarrow s)$ is a theorem of propositional logic, where s is any propositional formula. What does this result have to say about the importance of *consistency* for sets of propositional formulas?

(-) 6. Exercise 2.10 from the lectures: Show that if H is consistent and $H \vdash t$, then $H \cup \{t\}$ is also consistent.

(-) 7. Write down a deduction of $(p \Rightarrow q)$ from $\{\neg p\}$ (hint: use the result of question 5).

(-) 8. As claimed in the proof of the Adequacy Theorem in the lectures, show that $\{\neg s\} \vdash (s \Rightarrow t)$ (hint: use the result of question 5).

9. Let P be a countable set of primitive propositions. Show that the set $\mathcal{L}(P)$ of propositional formulas over P is

   (i) countable;

   (ii) recursively enumerable.

   (Of course, by question 1, it would suffice to show (ii) for both cases.)

10. Show that the set of tautologies of propositional logic is recursively enumerable (hint: use the Completeness Theorem).

11. For this question, it helps to consider a tree-like representation of propositional formulas.

    (i) Show that if there is a deduction of t from $H \cup \{s\}$ in n lines (that is, a deduction $t_1, \ldots, t_n = t$), then $(s \Rightarrow t)$ can be deduced from H in at most $3n + 2$ lines.

    (ii) Show that there is deduction of $\perp$ from $\{((p \Rightarrow q) \Rightarrow p), (p \Rightarrow \perp)\}$ in 16 lines (hint: use question 7).

    (iii) From (i) and (ii), calculate an upper bound for the length of a proof of the tautology of question 2 (iii).

(+) 12. Let t be a propositional formula not involving the symbol $\perp$ and let $t' = t[\perp/p]$ be the formula obtained from t by substituting $\perp$ for all occurrences of a particular propositional variable p in t. Suppose that t' is a tautology but t it not.

    (i) Show that any proof of t' in propositional logic must involve an instance of the third axiom (**T**).

    (ii) Does this remain true if t is allowed to contain occurrences of $\perp$?

(+) 13. Let $t_1, t_2, \ldots$ be propositional formulas such that, for every valuation v, there exists n with $v(t_n) = 1$. Use the Compactness Theorem to show that in fact we may bound the value of n: there must be an $N \geq 1$ such that, for every valuation v, there exists $n \leq N$ with $v(t_n) = 1$.

(+) 14. From the proof of Lemma 2.11 in the lectures, explain why the set $H_{\max} := \bigcup_{i=0}^{\infty} T_i$ is consistent.

15. Consider the following theorem.

    Theorem (Compactness Theorem II)**.** *Let $H \subseteq \mathcal{L}(P)$ and $t \in \mathcal{L}(P)$. If $H \vDash t$ then there is a finite $H' \subseteq H$ such that $H' \vDash t$.*

    Show that this theorem is equivalent to the version of the Compactness Theorem we stated in the lectures; that is to say, show that from the Compactness Theorem we can conclude Compactness Theorem II and vice versa. What does Compactness Theorem II tell us about the existence of proofs in propositional logic?