Lecture 2: support

Name permutations

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- $\mathbf{Perm}\,\mathbb{A}$ = group of finite permutations of $\mathbb{A}$ (π, π',…)
  - each π is a bijection $\mathbb{A} \cong \mathbb{A}$ (injective and surjective function)
  - group: multiplication is composition of functions $\pi' \circ \pi$; identity is identity function ι; inverses are inverse functions π^{-1}.
  - π finite means: $\{a \in \mathbb{A} \mid \pi(a) \neq a\}$ is finite.

Quiz: if $\mathbb{A} = \{0, 1, 2, 3, \ldots\}$, are these maps in $\mathbf{Perm}\,\mathbb{A}$?

- $0 \mapsto 1 \mapsto 0, k \mapsto k$ (for all $k \geq 2$)
- $2k \mapsto 2k + 1 \mapsto 2k$ (for all $k \geq 0$)
- $0 \mapsto 1 \mapsto 2 \mapsto 3 \mapsto \cdots$

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- $0 \mapsto 1 \mapsto 0, k \mapsto k$ (for all $k \geq 2$) is in $\mathbf{Perm}\,\mathbb{A}$
- $2k \mapsto 2k+1 \mapsto 2k$ (for all $k \geq 0$) is not finite
- $0 \mapsto 1 \mapsto 2 \mapsto 3 \mapsto \cdots$ is not a bijection

Transposition

$(a\ b) \in \mathbf{Perm}\,\mathbb{A}$ is the function mapping a to b, b to a and fixing all other names.

Theorem: every $\pi \in \mathbf{Perm}\,\mathbb{A}$ is equal to
$$(a_1\ b_1) \circ \cdots \circ (a_n\ b_n)$$
for some a_i & b_i (with $\pi\, a_i \neq a_i \neq b_i \neq \pi\, b_i$).

Proof. . .

Actions

An $\mathbf{Perm}\,\mathbb{A}$-action on a set X is a function

$$(-)\cdot(-) : \mathbf{Perm}\,\mathbb{A} \times X \to X$$

satisfying for all $\pi, \pi' \in \mathbf{Perm}\,\mathbb{A}$ and $x \in X$

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Simple example: action of $\mathbf{Perm}\,\mathbb{A}$ on $\mathbb{A}$ given by function application, $\pi \cdot a = \pi\, a$.

$$\begin{aligned} &\pi' \cdot (\pi \cdot a) = \pi'(\pi\, a) = (\pi' \circ \pi)\, a \\ &\iota \cdot a = \iota\, a = a \end{aligned}$$

Running example

Action of **Perm** $\mathbb{A}$ on set of ASTs for λ-terms

$$Tr \triangleq \{t ::= \mathtt{V}\,a \mid \mathtt{A}(t,t) \mid \mathtt{L}(a,t)\}$$

$$\begin{array}{rcl} \pi \cdot \mathtt{V}\,a & = & \mathtt{V}(\pi\,a) \\ \pi \cdot \mathtt{A}(t,t') & = & \mathtt{A}(\pi \cdot t, \pi \cdot t') \\ \pi \cdot \mathtt{L}(a,t) & = & \mathtt{L}(\pi\,a, \pi \cdot t) \end{array}$$

This respects α-equivalence $=_\alpha$ [lecture 1]

$$t =_\alpha t' \Rightarrow \pi \cdot t =_\alpha \pi \cdot t' \quad \text{(Exercise)}$$

and so induces an action on set of λ-terms $\Lambda \triangleq \{[t]_\alpha \mid t \in Tr\}$:

$$\pi \cdot [t]_\alpha = [\pi \cdot t]_\alpha$$

Nominal sets

are sets $\boldsymbol{X}$ with with a $\mathbf{Perm}\,\mathbb{A}$-action satisfying

Finite support property: for each $x \in X$, there is a finite subset $A \subseteq \mathbb{A}$ that supports x, in the sense that for all $\pi \in \mathbf{Perm}\,\mathbb{A}$

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E.g. $\mathbb{A}$ is a nominal set—each $a \in \mathbb{A}$ is supported by any finite A containing a.

Tr and Λ are nominal sets—any finite A containing all the variables occurring (free, binding, or bound) in $t \in Tr$ supports t and (hence) $[t]_\alpha$.

Discrete nominal set

determined by a set S has permutation action

$$\pi \cdot x \triangleq x \qquad (x \in S)$$

Each $x \in S$ is supported by $\emptyset$.

Trivial, but useful: 'ordinary' sets are included in nominal sets.

Support of sets of names

$\mathbf{Perm}\,\mathbb{A}$ acts of sets of names $S \subseteq \mathbb{A}$ pointwise:
$\pi \cdot S \triangleq \{\pi\, a \mid a \in S\}$.

What is a support for the following sets of names?

- $S_1 \triangleq \{a\}$

- $S_2 \triangleq \mathbb{A} - \{a\}$

- $S_3 \triangleq \{a_0, a_2, a_4, \ldots\}$, supposing $\mathbb{A} = \{a_0, a_1, a_2, \ldots\}$

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  Answer: $\{a_0, a_2, a_4, \ldots\}$ is a support, and so is $\{a_1, a_3, a_5, \ldots\}$—but there is no finite support. S_3 does not exist in the 'world of nominal sets' (in that world $\mathbb{A}$ is infinite, but not enumerable—see later).

Lemma. In any nominal set X, every $x \in X$ possesses a smallest (wrt $\subseteq$) finite support, written $\mathit{supp}\, x$.

Proof. Suffices to show that if finite subsets A_1 and A_2 support x, then so does $A_1 \cap A_2$.

Lemma. In any nominal set X, every $x \in X$ possesses a smallest (wrt $\subseteq$) finite support, written $supp\,x$.

Proof. Suppose finite subsets A_1 and A_2 support x.

For any $a, a' \in \mathbb{A} - (A_1 \cap A_2)$ with $a \neq a'$,
claim that $(a\ a') \cdot x = x$.

Since every π fixing the elements of $A_1 \cap A_2$ can be expressed as a composition of transpositions $(a\ a')$ with $a, a' \in \mathbb{A} - (A_1 \cap A_2)$ and $a \neq a'$, we have $\pi \cdot x = x$. □

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For any $a, a' \in \mathbb{A} - (A_1 \cap A_2)$ with $a \neq a'$,
claim that $(a\ a') \cdot x = x$.

Pick any $a'' \in \mathbb{A} - (A_1 \cup A_2 \cup \{a, a'\})$ (infinite, hence non-empty). Note that

$$(a, a'' \notin A_1) \vee (a, a'' \notin A_2)$$

so $(a\ a'') \cdot x = x$. Similarly $(a'\ a'') \cdot x = x$.

But $(a\ a') = (a\ a'') \circ (a'\ a'') \circ (a\ a'')$.

So $(a\ a') \cdot x = x$. □claim

Free variables via support

Recall that $\Lambda = \{[t]_\alpha \mid t \in Tr\}$ has a $\mathbf{Perm}\,\mathbb{A}$-action satisfying $\pi \cdot [t]_\alpha = [\pi \cdot t]_\alpha$

Fact: for any $[t]_\alpha \in \Lambda$, $supp([t]_\alpha)$ is the finite set $\mathbf{fv}\,t$ of free variables of any representative AST t.

$$
\begin{aligned}
\mathbf{fv}(\mathtt{V}\,a) &= \{a\} \\
\mathbf{fv}(\mathtt{A}(t,t')) &= (\mathbf{fv}\,t) \cup (\mathbf{fv}\,t') \\
\mathbf{fv}(\mathtt{L}(a,t)) &= (\mathbf{fv}\,t) - \{a\}
\end{aligned}
$$

(Exercise)