L108: Categorical theory and logic
Exercise sheet 4

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1. Let X be a set. We consider the partial order $P(X)$ of subsets of X as a category, with a morphism $S \to T$ if S is a subset or equal to T. We can think of the objects of $P(X)$ as predicates on X.

   (a) Consider another set Y and consider the function $\pi_1^* : P(X) \to P(X \times Y)$ given by $\pi_1^*(S) = (S \times Y)$. Show that it is a functor between categories.

   (b) Let $\exists_{\pi_1} : P(X \times Y) \to P(X)$ be the function given by

   $$\exists_{\pi_1}(S) \stackrel{\text{def}}{=} \{x \in X \mid \exists y \in Y.(x, y) \in S\}.$$

   Show that $\exists_{\pi_1}$ is a functor and show that π_1^* is right adjoint to $\exists_{\pi_1}$.

   (c) Show that $\pi_1^* : P(X) \to P(X \times Y)$ has a right adjoint (a functor $P(X \times Y) \to P(X)$).

2. Let X be a set. The 'slice category' $\mathbf{Set}/X$ is defined as follows. The objects are pairs (P, f) where P is a set and $f : P \to X$ is a function. A morphism $(P, f) \to (Q, g)$ is a function $h : P \to Q$ such that $f = gh$.

   (We think of an object of the slice category as a proof-relevant predicate: it comprises a set of proofs P and a function $f : P \to X$ saying which elements of X the proofs are about. For any $x \in X$, we have a set $\{p \mid f(p) = x\}$ of proofs which is empty if the predicate is false for x.)

   (a) Finish showing that $\mathbf{Set}/X$ is indeed a category.

   (b) Show that $\mathbf{Set}/X$ has products.

   (c) Let Y be a set. Define a functor $\Sigma_{\pi_1} : \mathbf{Set}/(X \times Y) \to \mathbf{Set}/X$ which acts on objects by $\Sigma_{\pi_1}(P, f) \stackrel{\text{def}}{=} (P, \pi_1 \cdot f)$.

   (d) Show that $\Sigma_{\pi_1} : \mathbf{Set}/(X \times Y) \to \mathbf{Set}/X$ has a right adjoint, a functor ${\pi_1}^* : \mathbf{Set}/X \to \mathbf{Set}/(X \times Y)$.

   (e) Show that ${\pi_1}^* : \mathbf{Set}/X \to \mathbf{Set}/(X \times Y)$ has a right adjoint (which will be a functor $\mathbf{Set}/(X \times Y) \to \mathbf{Set}/X$).