L108: Categorical theory and logic
Exercise sheet 3

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1. Describe a functor $\mathit{Pow} : \mathbf{Set} \to \mathbf{Monoid}$, which acts on objects as follows: a set X is taken to the monoid $(\mathcal{P}_{\mathrm{fin}}(X), \cup, \emptyset)$, where $\mathcal{P}_{\mathrm{fin}}(X)$ is set of finite subsets of X.

2. Recall that the free monoid on a set X is the monoid of lists of elements of X, $(\mathit{List}(X), @, [])$. Recall that this extends to a functor $\mathit{List} : \mathbf{Set} \to \mathbf{Monoid}$.

   Given a list $[x_1, \dots, x_n]$ in $\mathit{List}(X)$, we can construct the finite set $\{x_1, \dots, x_n\}$ of all the elements of X that appear in the list. In this way, for each set X we have a function $\phi_X : \mathit{List}(X) \to \mathit{Pow}(X)$.

   (a) Show that each ϕ_X is a homomorphism between monoids.

   (b) Show that the family $\{\phi_X : \mathit{List}(X) \to \mathit{Pow}(X)\}_{X \in \mathbf{Set}}$ is a natural transformation $\mathit{List} \Rightarrow \mathit{Pow} : \mathbf{Set} \to \mathbf{Monoid}$.

   (c) Define a function $\mathit{Pow}(\mathbb{N}) \to \mathit{List}(\mathbb{N})$. Is it a homomorphism between monoids?

   (d) Let $U : \mathbf{Monoid} \to \mathbf{Set}$ be the forgetful functor. So $(U \circ \mathit{List}) : \mathbf{Set} \to \mathbf{Set}$ is the functor that takes a set to the lists over that set.
   Can you define a natural transformation $U \circ \mathit{Pow} \Rightarrow U \circ \mathit{List} : \mathbf{Set} \to \mathbf{Set}$?

3. For any set X we have a structure $\mathit{RevList}(X) = (\mathit{List}(X), ⓐ, [])$ where

$$[x_1, \dots, x_m] ⓐ [y_1, \dots, y_n] \stackrel{\text{def}}{=} [y_1, \dots, y_n, x_1, \dots x_m].$$

   (a) Show that $\mathit{RevList}(X)$ is always a monoid.

   (b) Show that the 'reverse' function $\mathit{rev}_X : \mathit{List}(X) \to \mathit{RevList}(X)$ is a homomorphism between monoids.

   (c) Show that the family of functions $\{\mathit{rev}_X : \mathit{List}(X) \to \mathit{RevList}(X)\}_{X \in \mathbf{Set}}$ is a natural transformation $\mathit{List} \Rightarrow \mathit{RevList} : \mathbf{Set} \to \mathbf{Monoid}$.

4. This question is about the free monoid functor, $\mathit{List} : \mathbf{Set} \to \mathbf{Monoid}$, the finite powerset functor, $\mathit{Pow} : \mathbf{Set} \to \mathbf{Monoid}$, and the forgetful functor, $U : \mathbf{Monoid} \to \mathbf{Set}$.

   Can you find interesting natural transformations of the following kinds?

   (a) $\mathrm{Id}_{\mathbf{Set}} \Rightarrow U \circ \mathit{List} : \mathbf{Set} \to \mathbf{Set}$
   (b) $\mathrm{Id}_{\mathbf{Set}} \Rightarrow U \circ \mathit{Pow} : \mathbf{Set} \to \mathbf{Set}$
   (c) $U \circ \mathit{List} \circ U \circ \mathit{List} \Rightarrow U \circ \mathit{List} : \mathbf{Set} \to \mathbf{Set}$
   (d) $U \circ \mathit{Pow} \circ U \circ \mathit{Pow} \Rightarrow U \circ \mathit{Pow} : \mathbf{Set} \to \mathbf{Set}$
   (e) $\mathit{List} \circ U \Rightarrow \mathrm{Id}_{\mathbf{Monoid}} : \mathbf{Monoid} \to \mathbf{Monoid}$
   (f) $\mathit{Pow} \circ U \Rightarrow \mathrm{Id}_{\mathbf{Monoid}} : \mathbf{Monoid} \to \mathbf{Monoid}$

5. Recall the category of arrows, $\hat{\Sigma}$: the objects are functions between sets, and the morphisms are commuting squares. What functors can you find between the category of arrows and the category of sets (in either direction)? What natural transformations can you find between the functors?

6. Recall from Sheet 2 the category $\mathbb{F}$ whose objects are natural numbers (considered as sets) and whose morphisms are functions. Show that $\mathbb{F}^{\mathrm{op}}$ is a free category with finite products over one base type. What is the relationship between $\mathbb{F}^{\mathrm{op}}$ and the syntactic category for the type theory of products (**Syn**), if there is only one base type?

7. Let $\mathcal{C}$ be the category whose objects are monoids, and where a morphism $(X, \cdot_X, i_X) \to (Y, \cdot_Y, i_Y)$ is a function $X \to Y$ — it need not respect the monoid structure. Composition in $\mathcal{C}$ is just the usual composition of functions.

   Let $\mathcal{D}$ be the category whose objects are non-empty sets and whose morphisms are functions. Composition is just the usual composition of functions.

   Show that the categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent.