**L108: Category theory and logic**
**Exercise sheet 1**

**Sam Staton**
sam.staton@cl.cam.ac.uk

Revised: October 22, 2012, 11:24.

1. An object $X$ in a category is an *initial object* if for any object $Y$ there is a unique morphism $X \to Y$.
   (a) Does the category of sets have any initial objects?
   (b) Does the category of monoids have any initial objects?
   (c) State and prove a theorem making precise the following informal statement: "initial objects are unique up-to unique isomorphism".

2. Let $\mathcal{C}$ be a category. The dual of $\mathcal{C}$ is a category $\mathcal{C}^{\mathrm{op}}$ with the same objects as $\mathcal{C}$ and where $\mathcal{C}^{\mathrm{op}}(A, B) = \mathcal{C}(B, A)$.
   (a) Complete the definition of the category $\mathcal{C}^{\mathrm{op}}$.
   (b) Give an example of a finite category which is not the same as its dual (up to isomorphism).
   (c) Give an example of a finite category that is the same as its dual.
   (d) Show that a category $\mathcal{C}$ has an initial object if and only if $\mathcal{C}^{\mathrm{op}}$ has a terminal object.

3. Let $X$ be a set. Define a "monoid over $X$" to be a structure $(Y, \cdot_Y, i_Y, f)$ where $(Y, \cdot_Y, i_Y)$ is a monoid and $f : X \to Y$ is a function.
   (a) Describe a category whose objects are monoids over $X$.
   (b) Does this category have an initial object?
   (c) State and prove a theorem making precise the following statement: "free monoids are unique up-to unique isomorphism".