Rule Instances

- *Rule instances* are pairs

$$(X/y)$$

where X is a set of *premises* and y is a *conclusion*.

Example Let U be a set.

$$\frac{}{(a,a)} \quad a \in U \qquad \frac{(a,b)}{(b,a)} \qquad \frac{(a,b) \quad (b,c)}{(a,c)}$$

Rule Instances

closed

- *Rule instances* are pairs

$$(X/y)$$

where X is a set of *premises* and y is a *conclusion*.

- A set of rule instances R specifies a way to build a set:

> Each rule instance (X/y) in R, stipulates that
>
> if all the elements of X are in the set then so is y.

For the example, a set $S \subseteq U \times U$ is closed whenever:

(1) $\forall a \in U.\ (a,a) \in S$.

& (2) $\forall (a,b) \in S.\ (b,a) \in S$

& (3) $\forall (a,b),(b,c) \in S.\ (a,c) \in S$

iff S is an equivalence relation

Rule Instances

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- A set of rule instances R specifies a way to build a set:

  Each rule instance (X/y) in R, stipulates that
  if all the elements of X are in the set then so is y.

- There is a least set with the above property! We denote it I_R and called it the set inductively defined by the rule instances R.

Closed Sets

Q is R-closed

whenever $\forall (X/y) \in R.\quad X \subseteq Q \Rightarrow y \in Q$

Let Q_1 & Q_2 be closed. Claim: $Q_1 \cap Q_2$ is closed.

Assume $(X/y) \in R$ and suppose $X \subseteq Q_1 \cap Q_2$.

We need show that $y \in Q_1 \cap Q_2$

Since $X \subseteq Q_1 \cap Q_2$ we have $X \subseteq Q_1$ and since Q_1 is closed

$y \in Q_1$. Analogously $X \subseteq Q_2$ implies $y \in Q_2$. Hence

$y \in Q_1 \cap Q_2$.

This argument works for any family of closed sets.

In particular we can consider the family of all closed sets.

Inductively-Defined Sets

$I_R \stackrel{def}{=} \bigcap \{Q \mid Q \text{ is } R\text{-closed}\}$

(1) I_R is R-closed

(2) For all R-closed sets Q, $I_R \subseteq Q$.

Gives a way of exhibiting elements of I_R.

To show $I_R \subseteq Q$
show instead
Q is R-closed

A proof principle

Consider $Q = \{x \in I_R \mid P(x)\} \subseteq I_R$

General Principle of Rule Induction

For I_R the set inductively defined by a set of rule instances R, if

for each rule instance (X/y) in R,

$$\big(\forall x \in X.\, x \in \mathrm{I}_R \;\&\; P(x)\big) \Rightarrow P(y)$$

equivalent to

Q is R-closed

$\Downarrow$

then

$P(z)$ holds for all $z \in \mathrm{I}_R$

equivalent

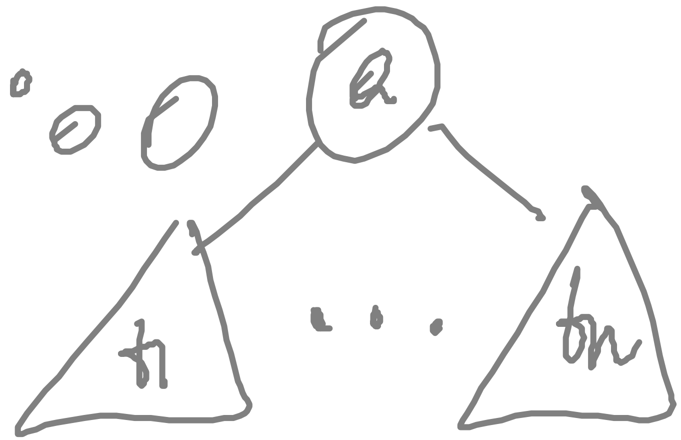
$I_R \subseteq Q$

Finitary Rule Instances

They have a finite number of premises.

Example: The rules for finitely branching trees.

FBT: $$\frac{t_1,\dots,t_n}{a[t_1,\dots,t_n]} \qquad n\in\mathbb{N}_0,\ a\in A$$



Take $A = [2]$

$I_{\underline{FBT}}$ $\sim$ is $\underline{FBT}$-closed

That is

$\forall n \in \mathbb{N}_0.\ t_1, \dots, t_n \in I_{\underline{FBT}}.$

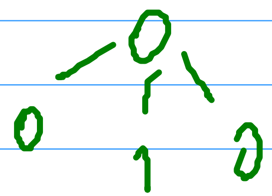
$$\Rightarrow \quad 0[t_1, \dots, t_n] \in I_{\underline{FBT}}$$

$$\wedge$$

$$1[t_1 \dots t_n] \in I_{\underline{FBT}}$$

Then $I_{\underline{FBT}}$ contains

$0[\,]$, $1[\,]$, $0[0[\,], 1[\,], 0[\,]]$, ...

0 , 1 ,  , ...

Finitary Closure

For I_R the set inductively defined by a set of *finitary* rule instances R,

for each axiom $(\ /y)$,

$y \in I_R$

and

for each rule instance $(\{x_1, \ldots, x_n\}/y)$,

if $x_i \in I_R$ for all $1 \leq i \leq n$ then $y \in I_R$

Finitary Principle of Rule Induction

For I_R the set inductively defined by a set of *finitary* rule instances R,

if

for each axiom $(\ /y)$,

$P(y)$ holds

and

for each rule instance $(\{x_1, \ldots, x_n\}/y)$,

$x_i \in I_R$ and $P(x_i)$ for all $1 \leq i \leq n$ implies $P(y)$

then

$P(z)$ holds for all $z \in I_R$

$$\frac{}{\underline{zero}} \qquad \frac{n}{\underline{succ}(n)}$$

Examples:

- Principle of mathematical induction

  If

  $P(\mathtt{zero})$ holds

  and

  for all $n \in I_{\mathbf{Nat}}$, $P(n) \Rightarrow P\big(\mathtt{succ}(n)\big)$

  then

  $P(k)$ holds for all $k \in I_{\mathbf{Nat}}$

$$\frac{}{\varepsilon} \qquad \frac{s}{a.s}\ \text{character } a$$

- Principle of induction for strings

If

$P(\varepsilon)$ holds

and

for each character a,

for all $s \in I_{\mathbf{String}}$, $P(s) \Rightarrow P(a.s)$

then

$P(w)$ for all $w \in I_{\mathbf{String}}$

- Principle of induction for Boolean propositions.

If

$P(a)$ holds, for all propositional variables a $\qquad \dfrac{}{a}$

and

$P(T)$ holds $\qquad \dfrac{}{T} \qquad \dfrac{}{F}$

and

$P(F)$ holds $\qquad \dfrac{A \quad B}{A \wedge B} \qquad \dfrac{A \quad B}{A \vee B}$

and

for all $A, B \in I_{\mathbf{BoolProp}}$, $P(A)$ & $P(B) \Rightarrow P(A \wedge B)$

and

for all $A, B \in I_{\mathbf{BoolProp}}$, $P(A)$ & $P(B) \Rightarrow P(A \vee B)$

and

for all $A \in I_{\mathbf{BoolProp}}$, $P(A) \Rightarrow P(\neg A)$ $\qquad \dfrac{A}{\neg A}$

then

$P(X)$ holds for all $X \in I_{\mathbf{BoolProp}}$

Exercise: Write the principle of induction for the inductively defined set of *regular expressions* given by the rules:

$$\frac{}{a}\ a \text{ a symbol} \qquad \frac{}{\varepsilon} \qquad \frac{}{\emptyset}$$

$$\frac{r \quad s}{r \mid s} \qquad \frac{r \quad s}{r.s} \qquad \frac{r}{r^*}$$

If $S \subseteq U \times U$ is $\underline{TC}(R)$-closed

(1) $\forall (a,b) \in R \,.\, (a,b) \in S$ (equivalently $R \subseteq S$)

& (2) $\forall (a,b),(b,c) \in S \,.\, (a,c) \in S$ (equivalently S is transitive)

Transitive Closure

The set of rule instances $\mathbf{TC}(R)$ for the transitive closure of a relation $R \subseteq U \times U$ is given by

$$\frac{}{(a,b)}\ (a,b) \in R \qquad\qquad \frac{(a,b) \quad (b,c)}{(a,c)}$$

Claim: $I_{\mathbf{TC}(R)} = R^+$

the least $\underline{TC}(R)$-closed set.

$R^+ =_{\text{def}} \bigcup_{n \in \mathbb{N}} R^n$