$\Gamma \vdash M : \tau$

$$\llbracket \Gamma \rrbracket \longrightarrow \llbracket \tau \rrbracket$$

$$\llbracket \Gamma \vdash M : \tau \rrbracket$$

or

$$\llbracket \Gamma \vdash M \rrbracket$$

or

$$\llbracket M \rrbracket$$

$$\Gamma \equiv (x_1 : \tau_1,\ x_2 : \tau_2,\ \ldots,\ x_n : \tau_n)$$

$$\llbracket \Gamma \rrbracket \cong \llbracket \tau_1 \rrbracket \times \llbracket \tau_2 \rrbracket \times \cdots \times \llbracket \tau_n \rrbracket$$

$\rho \in \llbracket \Gamma \rrbracket$ is a tuple $d_1, d_2, \ldots, d_n$ with $d_i \in \llbracket \tau_i \rrbracket$

$$[\![ \Gamma \vdash x : \tau ]\!]$$

$$[\![ x_1 : \tau_1 , \text{---} , x_n : \tau_n \vdash x_i : \tau_i ]\!] \qquad 1 \le i \le n$$

$$: [\![ \tau_1 ]\!] \times \cdots \times [\![ \tau_n ]\!] \longrightarrow [\![ \tau_i ]\!]$$

equiv. $\quad \rho \longmapsto \rho(x_i)$

$$(d_1, d_2, \text{---}, d_n) \longmapsto d_i$$

## Denotational semantics of PCF terms, I

$$[\![\Gamma \vdash \mathbf{0}]\!](\rho) \stackrel{\text{def}}{=} 0 \in [\![nat]\!]$$

$$[\![\Gamma \vdash \mathbf{true}]\!](\rho) \stackrel{\text{def}}{=} true \in [\![bool]\!]$$

$$[\![\Gamma \vdash \mathbf{false}]\!](\rho) \stackrel{\text{def}}{=} false \in [\![bool]\!]$$

$$[\![\Gamma \vdash x]\!](\rho) \stackrel{\text{def}}{=} \rho(x) \in [\![\Gamma(x)]\!] \qquad \big(x \in dom(\Gamma)\big)$$

## Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \perp \\ \perp & \text{if } [\![\Gamma \vdash M]\!](\rho) = \perp \end{cases}$$

$[\![\Gamma]\!] \longrightarrow nat = \mathbb{N}_\perp$

That $[\![M]\!]\rho = \perp$ gives that $[\![succ(M)]\!]\rho = \perp$ formalises the idea that succ(M) diverges whenever M diverges in the operational semantics

# Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

$$[\![\Gamma \vdash \mathbf{pred}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) - 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0, \bot \end{cases}$$

NB. If $[\![M]\!] = 0$ then $[\![\mathrm{pred}(M)]\!] = \bot$

a choice node in the operational semantics.

## Denotational semantics of PCF terms, II

These definitions give continuous functions.

$$\llbracket \Gamma \vdash \mathbf{succ}(M) \rrbracket(\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} \llbracket \Gamma \vdash M \rrbracket(\rho) + 1 & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) \neq \bot \\ \bot & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) = \bot \end{cases}$$

$$\llbracket \Gamma \vdash \mathbf{pred}(M) \rrbracket(\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} \llbracket \Gamma \vdash M \rrbracket(\rho) - 1 & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) > 0 \\ \bot & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) = 0, \bot \end{cases}$$

$$\llbracket \Gamma \vdash \mathbf{zero}(M) \rrbracket(\rho) \stackrel{\text{def}}{=} \begin{cases} \mathit{true} & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) = 0 \\ \mathit{false} & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) > 0 \\ \bot & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) = \bot \end{cases}$$

# Denotational semantics of PCF terms, III

$$[\![\Gamma \vdash \mathbf{if}\ M_1\ \mathbf{then}\ M_2\ \mathbf{else}\ M_3]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M_2]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{true} \\ [\![\Gamma \vdash M_3]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{false} \\ \bot & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \bot \end{cases}$$

# Denotational semantics of PCF terms, III

$$[\![\Gamma \vdash \mathbf{if}\ M_1\ \mathbf{then}\ M_2\ \mathbf{else}\ M_3]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M_2]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{true} \\ [\![\Gamma \vdash M_3]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{false} \\ \perp & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \perp \end{cases}$$

$$[\![\Gamma \vdash M_1\ M_2]\!](\rho) \stackrel{\text{def}}{=} \big([\![\Gamma \vdash M_1]\!](\rho)\big)\ ([\![\Gamma \vdash M_2]\!](\rho))$$

$$[\![\Gamma \vdash M_1(M_2) : \tau_2]\!]$$

$$\Gamma \vdash M_1 : \tau_1 \to \tau_2$$

$$\Gamma \vdash M_2 : \tau_1$$

$$[\![\Gamma]\!] \xrightarrow{[\![M_1]\!]} [\![\tau_1 \to \tau_2]\!] = ([\![\tau_1]\!] \to [\![\tau_2]\!])$$

$$[\![\Gamma]\!] \xrightarrow{[\![M_2]\!]} [\![\tau_1]\!]$$

$$[\![\Gamma \vdash M_1(M_2)]\!]\rho = [\![M_1]\!](\rho)\left([\![M_2]\!](\rho)\right)$$

$$[\![\Gamma \vdash \mathbf{fn}\, x. M : \tau_1 \to \tau_2]\!]$$

$$: [\![\Gamma]\!] \longrightarrow [\![\tau_1 \to \tau_2]\!] = \left([\![\tau_1]\!] \to [\![\tau_2]\!]\right)$$

<u>Claim</u>: This definition yields a continuous function.

By induction

$$[\![\Gamma, x:\tau_1 \vdash M : \tau_2]\!]$$

$$: \underbrace{[\![\Gamma, x:\tau_1]\!]}_{\substack{\cong \\ [\![\Gamma]\!] \times [\![\tau_1]\!]}} \longrightarrow [\![\tau_2]\!]$$

$$[\![\Gamma \vdash \mathbf{fn}\, x. M]\!](\underset{\in [\![\Gamma]\!]}{\rho})(\underset{\in [\![\tau_1]\!]}{d}) \overset{def}{=} \overbrace{[\![\Gamma, x \vdash M]\!](\rho, d)}^{\in [\![\tau_2]\!]}$$

$f : D \times E \to F \rightsquigarrow \hat{f} : D \to (E \to F)$

$$\hat{f}(d)(e) \underset{def}{=} f(d,e)$$

Currying

**CLAIM:** transforms continuous functions into continuous functions

Assume $f$ continuous

(1) Show $\hat{f}$ monotone:

$d \sqsubseteq d'$ in $D \overset{?}{\Longrightarrow} \hat{f}(d) \sqsubseteq \hat{f}(d')$ in $(E \to F)$

Suppose $d \sqsubseteq d'$ in $D$

Check $\hat{f}d \sqsubseteq \hat{f}d'$ in $(E \to F)$

or equivalently

$$\hat{f}(d)(e) \underset{\text{in } F}{\sqsubseteq} \hat{f}(d')(e) \quad \forall e \in E$$

But

$$\hat{f}(d)(e) = f(d,e)$$

&

$$\hat{f}(d')(e) = f(d',e)$$

Finally $d \sqsubseteq d' \Rightarrow (d,e) \sqsubseteq (d',e) \Rightarrow f(d,e) \sqsubseteq f(d',e)$

(2) Show $\hat{f}$ is continuous.

$$\hat{f}\left(\bigsqcup_n d_n\right) \overset{?}{=} \bigsqcup_n \hat{f}(d_n) \quad \text{in } (E \to F)$$

$$\text{iff} \quad \forall e \in E \quad \hat{f}\left(\bigsqcup_n d_n\right)(e) \overset{?}{=} \left(\bigsqcup_n \hat{f}(d_n)\right)(e)$$

$$f\left(\bigsqcup_n d_n, e\right) \qquad\qquad \bigsqcup_n \left(\hat{f}(d_n)(e)\right)$$

$$\bigsqcup_n f(d_n, e) \;=\!=\; \bigsqcup_n f(d_n, e)$$

## Denotational semantics of PCF terms, IV

$$
\begin{aligned}
&[\![\Gamma \vdash \mathbf{fn}\, x : \tau \,.\, M]\!](\rho) && (x \notin dom(\Gamma))\\
&\stackrel{\text{def}}{=} \lambda d \in [\![\tau]\!] \,.\, [\![\Gamma[x \mapsto \tau] \vdash M]\!](\rho[x \mapsto d])
\end{aligned}
$$

**NB:** $\rho[x \mapsto d] \in [\![\Gamma[x \mapsto \tau]]\!]$ is the function mapping $x$ to $d \in [\![\tau]\!]$ and otherwise acting like $\rho$.

$$\frac{\Gamma \vdash M : \tau \to \tau}{\Gamma \vdash \underline{fix}\, M : \tau}$$

$[\![M]\!] : [\![\Gamma]\!] \to ([\![\tau]\!] \to [\![\tau]\!])$

$[\![fix\, M]\!] : [\![\Gamma]\!] \longrightarrow [\![\tau]\!]$

## Denotational semantics of PCF terms, V

$$[\![\Gamma \vdash \mathbf{fix}(M)]\!](\rho) \stackrel{\text{def}}{=} \mathit{fix}([\![\Gamma \vdash M]\!](\rho))$$

Recall: $\underline{fix} : (D \to D) \to D$ is continuous

Recall that $\mathit{fix}$ is the function assigning least fixed points to continuous functions.

$[\![fix\, M]\!] = \underline{fix} \circ [\![M]\!]$ is continuous.

## Denotational semantics of PCF

**Proposition.** *For all typing judgements* $\Gamma \vdash M : \tau$*, the denotation*

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

*is a well-defined continous function.*

## Denotations of closed terms

For a closed term $M \in \mathrm{PCF}_\tau$, we get

$$[\![\emptyset \vdash M]\!] : [\![\emptyset]\!] \to [\![\tau]\!]$$

and, since $[\![\emptyset]\!] = \{\perp\}$, we have

$$\boxed{[\![M]\!] \stackrel{\text{def}}{=} [\![\emptyset \vdash M]\!](\perp) \in [\![\tau]\!] \qquad (M \in \mathrm{PCF}_\tau)}$$

Example $[\![x:\tau \vdash M_1 : \tau']\!] = [\![x:\tau \vdash M_2 : \tau']\!]$

$\Longrightarrow [\![\mathbf{fn}\, x:\tau.\, M_1 : \tau \to \tau']\!] = [\![\mathbf{fn}\, x:\tau.\, M_2 : \tau \to \tau']\!]$

## Compositionality

**Proposition.** *For all typing judgements* $\Gamma \vdash M : \tau$ *and* $\Gamma \vdash M' : \tau$, *and all contexts* $\mathcal{C}[-]$ *such that* $\Gamma' \vdash \mathcal{C}[M] : \tau'$ *and* $\Gamma' \vdash \mathcal{C}[M'] : \tau'$,

*if* $[\![\Gamma \vdash M]\!] = [\![\Gamma \vdash M']\!] : [\![\Gamma]\!] \to [\![\tau]\!]$

*then* $[\![\Gamma' \vdash \mathcal{C}[M]]\!] = [\![\Gamma' \vdash \mathcal{C}[M]]\!] : [\![\Gamma']\!] \to [\![\tau']\!]$

## Soundness

**Proposition.** *For all closed terms* $M, V \in \mathrm{PCF}_\tau$,

$$\textit{if } M \Downarrow_\tau V \textit{ then } [\![M]\!] = [\![V]\!] \in [\![\tau]\!] \ .$$

evaluation equality

$M_1(M_2) \Downarrow V$ Then $[\![M_1(M_2)]\!] \stackrel{?}{=} [\![V]\!]$

$$\frac{M_1 \Downarrow fn\, x.M \qquad M[M_2/x] \Downarrow V}{M_1\, M_2 \Downarrow V}$$

By induction $[\![M_1]\!] = [\![fn\, x.M]\!] = \lambda d.[\![M]\!](d)$ ①

substitution lemma

$$[\![M[M_2/x]]\!] = [\![V]\!]$$

$$[\![M_1(M_2)]\!] = [\![M_1]\!]([\![M_2]\!]) \stackrel{\text{by ①}}{=} [\![M]\!]([\![M_2]\!]) \stackrel{\text{by ②}}{=} [\![V]\!]$$

LEMMA: $[\![M[M_2/x]]\!] = [\![M]\!]([\![M_2]\!])$ ②

# Substitution property

**Proposition.** *Suppose that* $\Gamma \vdash M : \tau$ *and that* $\Gamma[x \mapsto \tau] \vdash M' : \tau'$, *so that we also have* $\Gamma \vdash M'[M/x] : \tau'$.

*Then,*

$$
\begin{aligned}
&[\![\Gamma \vdash M'[M/x]]\!](\rho) \\
&\quad = [\![\Gamma[x \mapsto \tau] \vdash M']\!]\big(\rho\big[x \mapsto [\![\Gamma \vdash M]\!]\big]\big)
\end{aligned}
$$

*for all* $\rho \in [\![\Gamma]\!]$.

substitution

application/composition

# Substitution property

**Proposition.** *Suppose that* $\Gamma \vdash M : \tau$ *and that* $\Gamma[x \mapsto \tau] \vdash M' : \tau'$*, so that we also have* $\Gamma \vdash M'[M/x] : \tau'$*.*

*Then,*

$$
\begin{aligned}
&[\![\Gamma \vdash M'[M/x]]\!](\rho) \\
&\quad = [\![\Gamma[x \mapsto \tau] \vdash M']\!]\big(\rho\big[x \mapsto [\![\Gamma \vdash M]\!]\big]\big)
\end{aligned}
$$

*for all* $\rho \in [\![\Gamma]\!]$*.*

---

In particular when $\Gamma = \emptyset$, $[\![\langle x \mapsto \tau \rangle \vdash M']\!] : [\![\tau]\!] \to [\![\tau']\!]$ and

$$
[\![M'[M/x]]\!] = [\![\langle x \mapsto \tau \rangle \vdash M']\!]([\![M]\!])
$$