Binary product of cpo's and domains

The product of two cpo's $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \ \& \ d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined by

$$(d_1, d_2) \sqsubseteq (d'_1, d'_2) \ \overset{\text{def}}{\Leftrightarrow} \ d_1 \sqsubseteq_1 d'_1 \ \& \ d_2 \sqsubseteq_2 d'_2 \ .$$

$$\frac{(x_1, x_2) \sqsubseteq (y_1, y_2)}{x_1 \sqsubseteq_1 y_1 \qquad x_2 \sqsubseteq_2 y_2}$$

Lubs of chains are calculated componentwise:

$$\bigsqcup_{n\geq 0}(d_{1,n}, d_{2,n}) = (\bigsqcup_{i\geq 0} d_{1,i}, \bigsqcup_{j\geq 0} d_{2,j}) \ .$$

If $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ are domains so is $(D_1 \times D_2, \sqsubseteq)$ and $\perp_{D_1 \times D_2} = (\perp_{D_1}, \perp_{D_2})$.

f monotone means for $\forall\, x, y \in (D \times E)$

$x \sqsubseteq_{D\times E} y \Rightarrow f(x) \sqsubseteq f(y)$

Equivalently $\forall\, (x_1, x_2), (y_1, y_2) \in D \times E$

$x_1 \sqsubseteq y_1$ & $x_2 \sqsubseteq y_2$ $\Rightarrow f(x_1, x_2) \sqsubseteq f(y_1, y_2)$

Proposition. *Let D, E, F be cpo's. A function $f : (D \times E) \to F$ is monotone if and only if it is monotone in each argument separately:*

monotonicity in the first argument

$$\forall d, d' \in D, e \in E.\, d \sqsubseteq d' \;\Rightarrow\; f(d, e) \sqsubseteq f(d', e)$$

$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \;\Rightarrow\; f(d, e) \sqsubseteq f(d, e').$$

monotonicity in the second argument.

Moreover, it is continuous if and only if it preserves lubs of chains in each argument separately:

$$f(\bigsqcup_{m\geq 0} d_m\,,\, e) = \bigsqcup_{m\geq 0} f(d_m, e)$$

$$f(d\,,\, \bigsqcup_{n\geq 0} e_n) = \bigsqcup_{n\geq 0} f(d, e_n).$$

Continuity in two arguments $f(\bigsqcup_n (d_n, e_n)) = \bigsqcup f(d_n, e_n)$

Continuity in 2 arguments $\Rightarrow$ Continuity in each argument.

Assume $f\left(\bigsqcup_n (d_n, e_n)\right) = \bigsqcup_n f(d_n, e_n)$

Show $f(d, \bigsqcup_n e_n) \stackrel{?}{=} \bigsqcup_n f(d, e_n)$

$\| \quad \| \text{ by assumption}$

$f\left(\bigsqcup_n (d, e_n)\right)$

$f(\bigsqcup_n d, \bigsqcup_n e_n)$

$\boxed{\underline{\text{NB}}: \bigsqcup_n d = d}$

Continuity in each argument $\Rightarrow$ Continuity in two arguments.

Assume ...

Show $f(\bigsqcup_n (d_n, e_n)) \overset{?}{=} \bigsqcup_n f(d_n, e_n)$

$$f(\bigsqcup_n d_n, \bigsqcup_n e_n) \overset{?}{=} \bigsqcup_m f(\bigsqcup_n d_n, e_m)$$

by cont in 2nd arg.

$\|$ by cont in the 1st arg.

$$\bigsqcup_m \bigsqcup_n f(d_n, e_m)$$

$$\bigsqcup_k f(d_k, e_k) = \text{Diag.}$$

- A couple of derived rules:

$$\frac{x \sqsubseteq x' \qquad y \sqsubseteq y'}{f(x,y) \sqsubseteq f(x',y')} \quad (f \text{ monotone})$$

$$\frac{}{f(\bigsqcup_m x_m, \bigsqcup_n y_n) = \bigsqcup_k f(x_k, y_k)}$$

Function cpo's and domains

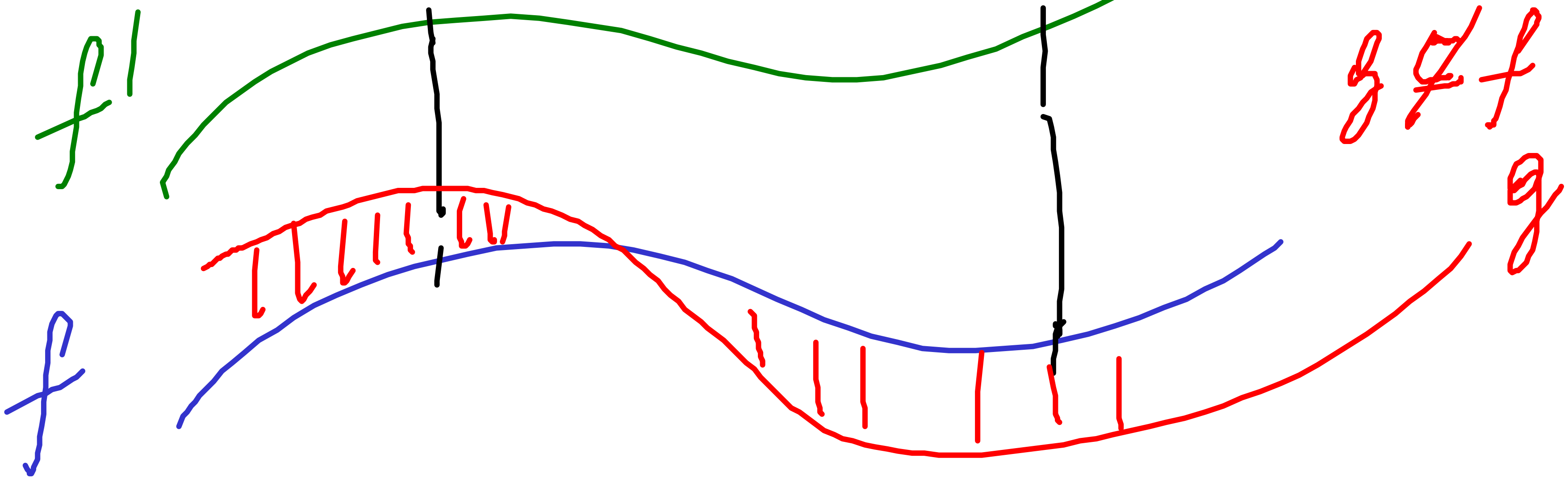
Given cpo's $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$(D \to E) \stackrel{\text{def}}{=} \{f \mid f : D \to E \text{ is a } \textit{continuous} \text{ function}\}$$

and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\Leftrightarrow} \forall d \in D \,.\, f(d) \sqsubseteq_E f'(d)$.

$f \not\sqsubseteq g$

$g \not\sqsubseteq f$



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- A derived rule:

$$\frac{f \sqsubseteq_{(D \to E)} g \qquad x \sqsubseteq_D y}{f(x) \sqsubseteq g(y)}$$

$$\dfrac{f \sqsubseteq g \Rightarrow fx \sqsubseteq gx \qquad \dfrac{x \sqsubseteq y}{gx \sqsubseteq gy}\ g \text{ mon}}{f(x) \sqsubseteq g(y)}$$

Show $((D \to E), \sqsubseteq)$ is a partial order

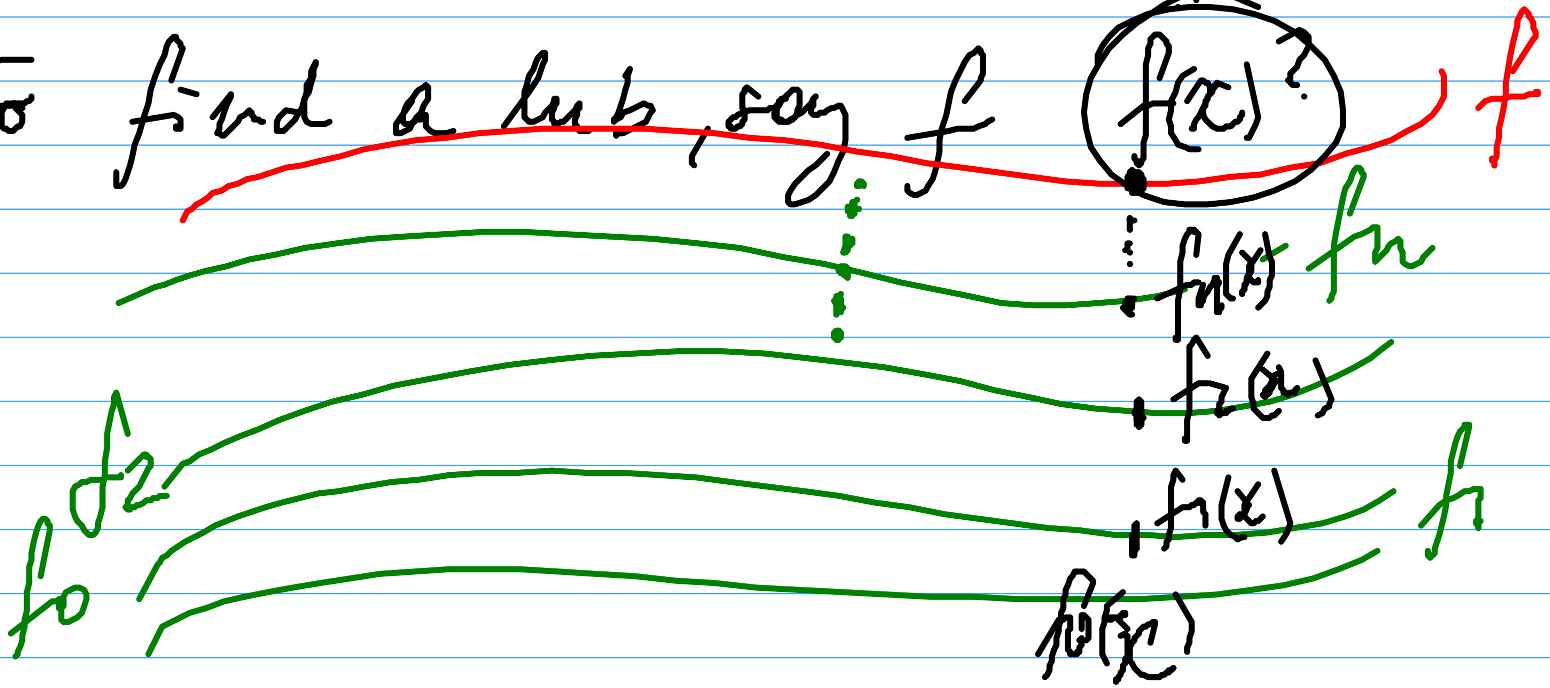
& has lubs of chains.

Given a chain

$f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots$ in $(D \to E)$

Take $\bigsqcup_n f_n(x)$

need to find a lub, say f



Define f to be given by $f(x) = \bigsqcup_n f_n(x)$.

- f is a function from $D \to E$.

$\boxed{?}$ If $f \in (D \to E)$? I.e is f cont.?

Show $f(\bigsqcup_k x_k) \stackrel{?}{=} \bigsqcup_k f(x_k)$ lub

$$\bigsqcup_n f_n(\bigsqcup_k x_k) \underbrace{=}_{\text{fn continuous}} \bigsqcup_n \bigsqcup_k f_n(x_k)$$

$\sim$ diag.

(1) $f_n \overset{?}{\sqsubseteq} f$ iff $\forall x.\ f_n(x) \sqsubseteq f(x)$

$$f(x) = \bigsqcup_n f_n(x)$$

(2) $f_n \sqsubseteq g \Rightarrow \underbrace{f \sqsubseteq g}$

iff $f(x) \sqsubseteq g(x)\ \forall x$

$$f(x) = \bigsqcup_n f_n(x)$$

$f_n \sqsubseteq g$

$\Longrightarrow \forall x\ \ f_n(x) \sqsubseteq g(x) \Longrightarrow \bigsqcup_n f_n(x) \sqsubseteq g(x).$

Lubs of chains are calculated 'argumentwise' (using lubs in E):

$$\bigsqcup_{n \geq 0} f_n \;=\; \lambda d \in D.\, \bigsqcup_{n \geq 0} f_n(d) \;.$$

If E is a domain, then so is $D \to E$ and $\perp_{D \to E}(d) = \perp_E$, all $d \in D$.

Lubs of chains are calculated 'argumentwise' (using lubs in E):

$$\bigsqcup_{n \geq 0} f_n \; = \; \lambda d \in D. \bigsqcup_{n \geq 0} f_n(d) \; .$$

- A derived rule:

$$\frac{}{(\bigsqcup_n f_n)(\bigsqcup_m x_m) = \bigsqcup_k f_k(x_k)}$$

If E is a domain, then so is $D \to E$ and $\perp_{D \to E}(d) = \perp_E$, all $d \in D$.

Continuity of composition

For cpo's D, E, F, the composition function

$$\circ : \big((E \to F) \times (D \to E)\big) \longrightarrow (D \to F)$$

defined by setting, for all $f \in (D \to E)$ and $g \in (E \to F)$,

$$g \circ f \;=\; \lambda d \in D.\, g\big(f(d)\big)$$

is continuous.

Continuity:

$$\bigsqcup_n (f_n \circ g_n) = \left(\bigsqcup_n f_n\right) \circ \left(\bigsqcup g_n\right)$$

By lemma

$$\bigsqcup_n (f \circ g_n) = f \circ \left(\bigsqcup_n g_n\right)$$

$$* \quad \bigsqcup_n (f_n \circ g) = \left(\bigsqcup_n f_n\right) \circ g$$

$$\text{iff} \quad \forall x \left(\bigsqcup_n (f_n \circ g)\right)(x) \overset{?}{=} \left(\left(\bigsqcup_n f_n\right) \circ g\right)(x)$$

$$\left(\bigsqcup_n (f_n \circ g)\right)(x) = \bigsqcup_n ((f_n \circ g)\, x) = \bigsqcup_n f_n (g\, x)$$

$$\left(\left(\bigsqcup_n f_n\right) \circ g\right)(x) = \left(\bigsqcup_n f_n\right)(g\, x) = \bigsqcup_n f_n (g\, x)$$

$fix(f) = \bigsqcup_n f^n(\perp)$.

Continuity of the fixpoint operator

Let D be a domain.

By Tarski's Fixed Point Theorem we know that each continuous function $f \in (D \to D)$ possesses a least fixed point, $\mathit{fix}(f) \in D$.

Proposition. *The function*

$$\mathit{fix} : (D \to D) \to D$$

is continuous.

fix is continuous

(1) Is monotone:

$f \sqsubseteq g \Rightarrow fix(f) \sqsubseteq fix(g)$

$f \bot \sqsubseteq g \bot$

$f f \bot \sqsubseteq f g \bot \sqsubseteq g g \bot$

f mon $\quad f \sqsubseteq g$

Assume $f \sqsubseteq g$

Show $f(\bot) \sqsubseteq g(\bot)$ $\quad f f \bot \sqsubseteq g g \bot \ldots$

$$\frac{\forall n \quad f^n(\bot) \sqsubseteq g^n(\bot)}{\bigsqcup_n f^n(\bot) \sqsubseteq \bigsqcup_n g^n(\bot)}$$

(2) $\quad \underline{\text{fix}}\left(\bigsqcup_n f_n\right) \stackrel{?}{=} \bigsqcup_n \underline{\text{fix}}(f_n)$

$$\underline{\text{fix}}\left(\bigsqcup_n f_n\right) = \bigsqcup_k \left(\bigsqcup_n f_n\right)^k(\bot) \qquad\qquad \bigsqcup_n \underline{\text{fix}}(f_n) = \bigsqcup_n \bigsqcup_\ell f_n^{\ell}(\bot)$$

$\bigsqcup_k \left(\bigsqcup_n f_n\right)^k(\bot) = \;\sim$ inductive lemma. $\qquad\qquad \bigsqcup_n \bigsqcup_\ell f_n^{\ell}(\bot) =$ lub diag.

$$\bigsqcup_k \bigsqcup_n f_n^k(\bot) = \bigsqcup_m f_m^m(\bot)$$

$$\left(\bigsqcup_n f_n\right)\bot = \bigsqcup_n f_n(\bot)$$

$$\left(\bigsqcup_n f_n\right)\left(\left(\bigsqcup_n f_n\right)\bot\right) = \left(\bigsqcup_n f_n\right)\left(\bigsqcup_n f_n(\bot)\right)$$

$$= \bigsqcup_k \left(\bigsqcup_n f_n\right)(f_k \bot) = \bigsqcup_k \bigsqcup_n f_n(f_k \bot) = \bigsqcup_\ell f_\ell^2(\bot)$$

Topic 4

Scott Induction

Scott's Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain D.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of f is in S, *i.e.* that

$$\mathit{fix}(f) \in S \ ,$$

it suffices to prove

$$\forall d \in D \ (d \in S \ \Rightarrow \ f(d) \in S) \ .$$

Idea $x \in S$ iff x satisfies the property one is interested in.

$$\frac{d \in S \Rightarrow f(d) \in S}{\underline{fix}(f) \in S}$$

$\bigsqcup_n f^n(\bot) \in S$

WANT

$\boxed{?}$ For which kind of property S can this be asserted?

$$\underline{fix}(f) = \bigsqcup_n f^n(\bot)$$

$$\Rightarrow f^n(\bot) \in S$$

$$f^2(\bot) \in S$$

$$\Uparrow$$

$$f(\bot) \in S$$

$$\Uparrow$$

$\bot \in S$ Assume

Chain-closed and admissible subsets

Let D be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D

$$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \left(\bigsqcup_{n \geq 0} d_n \right) \in S$$

If D is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of D and $\perp \in S$.

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$$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \Big(\bigsqcup_{n\geq 0} d_n\Big) \in S$$

If D is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of D and $\perp \in S$.

A property $\Phi(d)$ of elements $d \in D$ is called *chain-closed* (resp. *admissible*) iff $\{d \in D \mid \Phi(d)\}$ is a *chain-closed* (resp. *admissible*) subset of D.