$$\bigsqcup_n d_n^{(0)} \sqsubseteq \bigsqcup_n d_n^{(1)} \sqsubseteq \bigsqcup_n d_n^{(2)} \sqsubseteq$$

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
d_2^{(0)} & \sqsubseteq & d_2^{(1)} & \sqsubseteq & d_2^{(2)} & \sqsubseteq \cdots & d_2^{(n)} \\
\sqsubseteq & & \sqsubseteq & & \sqsubseteq & & \sqsubseteq \\
d_1^{(0)} & \sqsubseteq & d_1^{(1)} & \sqsubseteq & d_1^{(2)} & \sqsubseteq \cdots & d^{(n)} \\
\sqsubseteq & & \sqsubseteq & & \sqsubseteq & & \sqsubseteq \\
d_0^{(0)} & \sqsubseteq & d_0^{(1)} & \sqsubseteq & d_0^{(2)} & \sqsubseteq \cdots & d_0^{(n)}
\end{array}
\qquad \bigsqcup_m d_0^{(m)}$$

Fix the super index, say to m : $\bigsqcup_n d_n^{(m)}$

We can take

$$\bigsqcup_m \left(\bigsqcup_n d_n^{(m)} \right)$$

and

$$\bigsqcup_n \left(\bigsqcup_m d_n^{(m)} \right)$$

How do they compare?

There is also another chain to be considered namely the one on the diagonal

$$d_0^{(0)} \sqsubseteq d_1^{(1)} \sqsubseteq \cdots \sqsubseteq d_n^{(n)} \sqsubseteq \cdots$$

and so we can take

$$\bigsqcup_k d_k^{(k)}$$

<u>Claim:</u>

$$\| \quad \bigsqcup_m \left(\bigsqcup_n d_n^{(m)} \right)$$

Note:

$$\underbrace{d_n^{(m)} \underset{\substack{\forall n \\ \text{by lub1}}}{\sqsubseteq} \bigsqcup_n d_n^{(m)} \underset{\substack{\forall m \\ \text{by lub 1}}}{\sqsubseteq} \bigsqcup_m \bigsqcup_n d_n^{(m)}}$$

$$\Downarrow$$

$$\frac{\forall k \quad d_k^{(k)} \sqsubseteq \bigsqcup_m \bigsqcup_n d_n^{(m)}}{\bigsqcup_k d_k^{(k)} \sqsubseteq \bigsqcup_m \bigsqcup_n d_n^{(m)}} \ \text{by lub2}$$

Note: $d_k^{(k)} \sqsubseteq \bigsqcup_k d_k^{(k)}$

$$d_n^{m} \sqsubseteq d_{\max(m,n)}^{(\max(m,n))} \sqsubseteq \bigsqcup_k d_k^{(k)}$$

$$\forall m \forall n \quad d_n^{(m)} \sqsubseteq \bigsqcup_k d_k^{(k)}$$

lub 2

$$\forall m \quad \bigsqcup_n d_n^{(m)} \sqsubseteq \bigsqcup_k d_k^{(k)}$$

lub 2

$$\bigsqcup_m \bigsqcup_n d_n^{(m)} \sqsubseteq \bigsqcup_k d_k^{(k)}$$

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

   if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \ \sqsubseteq \ \bigsqcup_n e_n$.

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

   if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \ \sqsubseteq \ \bigsqcup_n e_n$.

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq y_n}{\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n} \quad (\langle x_n \rangle \text{ and } \langle y_n \rangle \text{ chains})$$

Diagonalising a double chain

Lemma. *Let D be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ ($m, n \geq 0$) satisfies*

$$m \leq m' \;\&\; n \leq n' \;\Rightarrow\; d_{m,n} \sqsubseteq d_{m',n'}. \qquad (\dagger)$$

Then

$$\bigsqcup_{n \geq 0} d_{0,n} \;\sqsubseteq\; \bigsqcup_{n \geq 0} d_{1,n} \;\sqsubseteq\; \bigsqcup_{n \geq 0} d_{2,n} \;\sqsubseteq\; \ldots$$

and

$$\bigsqcup_{m \geq 0} d_{m,0} \;\sqsubseteq\; \bigsqcup_{m \geq 0} d_{m,1} \;\sqsubseteq\; \bigsqcup_{m \geq 0} d_{m,3} \;\sqsubseteq\; \ldots$$

Lemma. *Let D be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ ($m, n \geq 0$) satisfies*

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Moreover

$$\bigsqcup_{m \geq 0} \left(\bigsqcup_{n \geq 0} d_{m,n} \right) = \bigsqcup_{k \geq 0} d_{k,k} = \bigsqcup_{n \geq 0} \left(\bigsqcup_{m \geq 0} d_{m,n} \right) .$$

our abstract notion of computable

Continuity and strictness

- If D and E are cpo's, the function f is continuous iff

  $D \to E$

  1. it is monotone, and
  2. it $\boxed{\text{preserves lubs of chains}}$, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

$D \quad \bigsqcup_n d_n = d$

$$f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E.$$

$d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq$

$D \xrightarrow{f} E$

$f(d) = f(\bigsqcup_n d_n)$

$e = \bigsqcup_n f(d_n)$

$f(d_0) \sqsubseteq f(d_1) \sqsubseteq \ldots \sqsubseteq f(d_n) \sqsubseteq$

continuity asserts the equality.

Continuity and strictness

- If D and E are cpo's, the function f is continuous iff

  1. it is monotone, and

  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

  $$f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E.$$

- If D and E have least elements, then the function f is strict iff $f(\perp) = \perp$.

Recall:
- needed fix
- with posets & monotone functions there the least prefixed point which is a fixed point.
- with domains & continuous functions fix always exist:

Tarski's Fixed Point Theorem

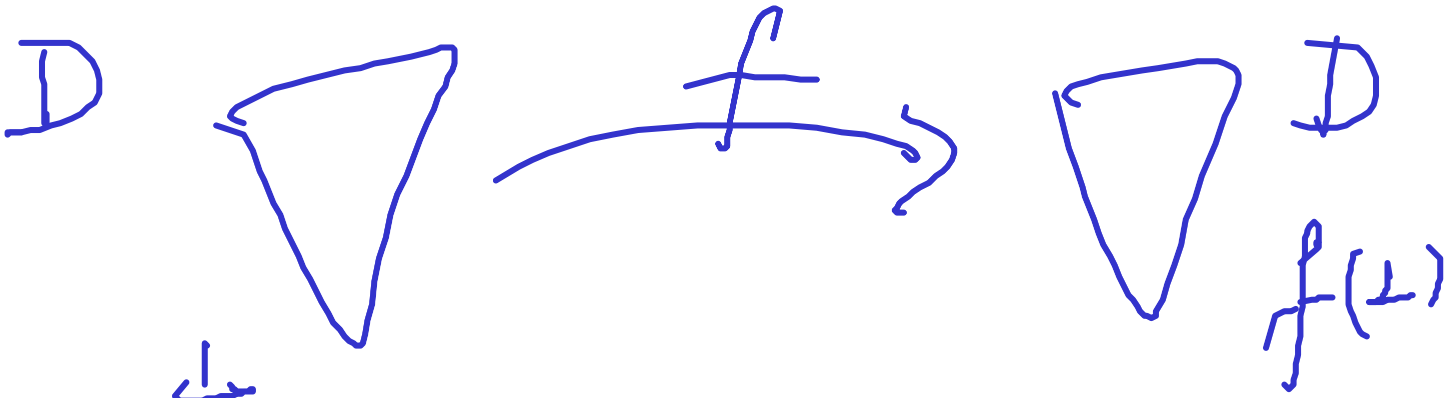
Let $f : D \to D$ be a continuous function on a domain D. Then

- f possesses a least pre-fixed point, given by

$$fix(f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

- Moreover, $fix(f)$ is a fixed point of f, *i.e.* satisfies $f(fix(f)) = fix(f)$, and hence is the least fixed point of f.

$\bot \sqsubseteq f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq \cdots \sqsubseteq f^n(\bot)$



<u>Claim</u> $\quad fix(f) = \bigsqcup_n f^n(\bot)$

is a least prefixed point.

(1) $\quad f(\underline{fix}\, f) = \underline{fix}(f)$

$$f\Big(\bigsqcup_{n\geq 0} f^n(\bot)\Big) = \bigsqcup_{n\geq 0} f(f^n(\bot)) \qquad \text{cont.}$$

$$= \bigsqcup_{n\geq 1} f^n(\bot)$$

$$= \underline{fix}(f)$$

(2) $f(x) \sqsubseteq x \implies \mathrm{fix}(f) \sqsubseteq x$

$$\perp \sqsubseteq x \implies f(\perp) \sqsubseteq f(x) \sqsubseteq x$$

$$\perp \sqsubseteq x \qquad f(\perp) \sqsubseteq x \qquad ff(\perp) \sqsubseteq x \qquad \cdots$$

$$\forall n \quad f^n(\perp) \sqsubseteq x$$

$$\bigsqcup_n f^n(\perp) = \underline{\mathrm{fix}}(f) \sqsubseteq x$$

$[\![\textbf{while } B \textbf{ do } C]\!]$

$$
\begin{aligned}
& [\![\textbf{while } B \textbf{ do } C]\!] \\
= {} & \mathit{fix}(f_{[\![B]\!],[\![C]\!]}) \\
= {} & \textstyle\bigsqcup_{n\geq 0} {f_{[\![B]\!],[\![C]\!]}}^{n}(\perp) \\
= {} & \lambda s \in \mathit{State}. \\
& \begin{cases}
[\![C]\!]^k(s) & \text{if } k \geq 0 \text{ is such that } [\![B]\!]([\![C]\!]^k(s)) = \mathit{false} \\
& \text{and } [\![B]\!]([\![C]\!]^i(s)) = \mathit{true} \text{ for all } 0 \leq i < k \\
\\
\text{undefined} & \text{if } [\![B]\!]([\![C]\!]^i(s)) = \mathit{true} \text{ for all } i \geq 0
\end{cases}
\end{aligned}
$$

$$[\![\text{while } B \text{ do } C]\!] \sqsubseteq [\![P]\!]$$

$$\|$$

$$\underline{\text{fix}}(f_{[\![B]\!],[\![C]\!]})$$

Scott Induction

$$\|$$

$$\bigsqcup_n f^n_{[\![B]\!],[\![C]\!]}(\bot) \sqsubseteq [\![P]\!]$$

we can use proof rules

can be lifted to $\underline{\text{fix}}$

(*) datatype

$D = \text{fn of } D \to D$

Topic 3

Constructions on Domains

In ML:

PRODUCT $*$

FUNCTION TYPES $\to$

ENUMERATED TYPES

INDUCTIVE (e.g. trees)

RECURSIVE (*)

datatypes

Every set can be made into a domain

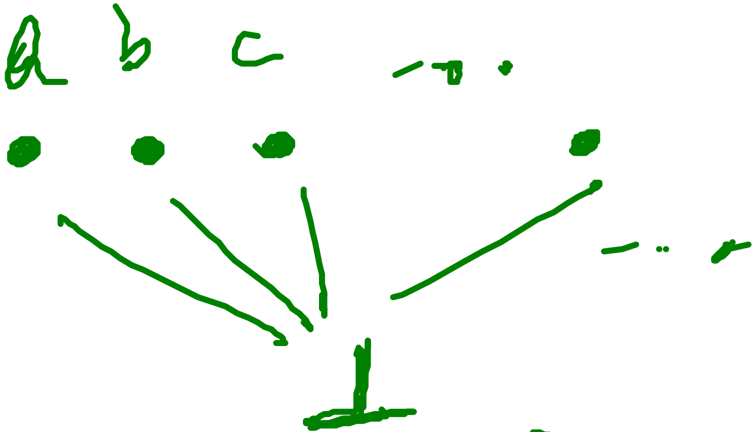
Discrete cpo's and flat domains

For any set X, the relation of equality

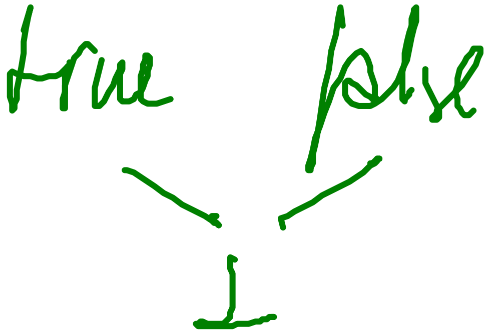
$$x \sqsubseteq x' \overset{\text{def}}{\Leftrightarrow} x = x' \qquad (x, x' \in X)$$

makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set X.

the natural way to order the set is by equality



$\mathbb{B}_\perp =$



Discrete cpo's and flat domains

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makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set X.

Let $X_\perp \overset{\text{def}}{=} X \cup \{\perp\}$, where $\perp$ is some element not in X. Then

$$d \sqsubseteq d' \overset{\text{def}}{\Leftrightarrow} (d = d') \vee (d = \perp) \qquad (d, d' \in X_\perp)$$

makes $(X_\perp, \sqsubseteq)$ into a domain (with least element $\perp$), called the flat domain determined by X.

Given domains $D_1 \stackrel{\sqsubseteq_1}{}$ & $D_2 \stackrel{\sqsubseteq_2}{}$

is there a natural construction for the <u>product type</u>

$$D_1 \times D_2 \quad \sqsubseteq \quad ?$$

$$D_1 \times D_2 \stackrel{def}{=} \{ (d_1, d_2) \mid d_1 \in D_1 \ \& \ d_2 \in D_2 \}$$

$$(d_1, d_2) \sqsubseteq (d_1', d_2')$$

$$\text{iff}_{def} \quad \underbrace{d_1 \sqsubseteq_1 d_1'}_{\text{in } D_1} \ \& \ \underbrace{d_2 \sqsubseteq_2 d_2'}_{\text{in } D_2}$$

Check that $(D_1 \times D_2, \sqsubseteq)$ is a domain.

(1) $\sqsubseteq$ is a partial order ✓

(2) We have a least element

$$\bot = (\bot_1, \bot_2) \in D_1 \times D_2$$

(3) We have lubs.

$$(x_0, y_0) \sqsubseteq (x_1, y_1) \sqsubseteq \cdots \sqsubseteq (x_n, y_n)$$

$$\bigsqcup_n (x_n, y_n) = \left(\bigsqcup_n x_n, \bigsqcup_n y_n\right).$$

Binary product of cpo's and domains

The product of two cpo's $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \;\&\; d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined by

$$(d_1, d_2) \sqsubseteq (d_1', d_2') \overset{\text{def}}{\Leftrightarrow} d_1 \sqsubseteq_1 d_1' \;\&\; d_2 \sqsubseteq_2 d_2' \;.$$

$$\frac{(x_1, x_2) \sqsubseteq (y_1, y_2)}{x_1 \sqsubseteq_1 y_1 \qquad x_2 \sqsubseteq_2 y_2}$$

Lubs of chains are calculated componentwise:

$$\bigsqcup_{n \geq 0} (d_{1,n}, d_{2,n}) = (\bigsqcup_{i \geq 0} d_{1,i}, \bigsqcup_{j \geq 0} d_{2,j}) \enspace .$$

If $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ are domains so is $(D_1 \times D_2, \sqsubseteq)$ and $\perp_{D_1 \times D_2} = (\perp_{D_1}, \perp_{D_2})$.