M.Phil in Advanced Computer Science 2011–12

Module R07: Introductory Logic (AM)

Exercises

Many of these exercises explore around points discussed in lectures; and are perhaps slightly nastier than an exam question.
[The symbol'*' against a question shows a rather more open-ended question.]

Question 1

(a) Define the idea of well-formed formula (wff) for propositional logic. "Well-formed" historically meant that formulae (seen as strings) were properly parethesised. What does it mean when we treat formulae as trees?

(b) What is a valuation? Explain what it means for a wff to be: valid, satisfiable, unsatisfiable. If wff A is satisfiable then is $\neg A$ unsatisfiable?

(c) Explain the difference between $\models$ and $\vdash$.

(d) Suppose we use two axiom schemes (where A, B, C stand for any wff and treating '$\rightarrow$' as right-associative): $A \rightarrow B \rightarrow A$ and $(A \rightarrow B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$ together with modus ponens.

- (i) Given any wff A, can $A \rightarrow A$ be deduced?
- (ii*) Can you find a valid wff which is not deducible?
- (iii) Can you find a wff which is deducible but not valid?
- (iv) Express this using words like 'soundness' and 'completeness'.

Question 2

(a) Summarise the ideas of first-order logic, including wffs, interpretations, model.

(b) Explain the notion of semantic entailment $\Gamma \models \phi$.

(c) Explain the difference between $\models$ and $\vdash$. To what extent are they identical or equivalent concepts

Question 3

(a) Explain the notions of compactness, a set of clauses being consistent and a set of clauses being satisfiable.

(b) Prove compactness for first-order logic assuming there is a sound and complete set of axioms and inference rules.

Question 4

(a) Explain what it means for a theory to be complete.

(b) Does a theory being complete mean that all its models are isomorphic? Give reasons.

(c) Can a theory have exactly two models – one infinte one and one finite one?

(d) To what extent can one write axioms which have models exactly when the model has an odd number of elements.

Question 5*

Attempt to axiomatise set theory. You should define a binary relation which models '$\in$', but prefer to define other relations such as '$\subseteq$' as abbreviations for wffs involving '$\in$'.

Question 6

(a) Explain the notion of deductive closure $Con\ \Gamma$ of a set of wffs Γ.

(b*) Author X defines a theory Θ to be axiomatisable when there is a decidable (a.k.a. recursive) set of wffs Γ such that $\Theta = Con\ \Gamma$, while author Y defines it to be axiomatisable when there is a recursively enumerable set of wffs Γ such that $\Theta = Con\ \Gamma$. Which, if either, author is more generous?

(c) Why do we not define a theory to be axiomatisable if it simply has a countable set of wffs Γ such that $\Theta = Con\ \Gamma$?