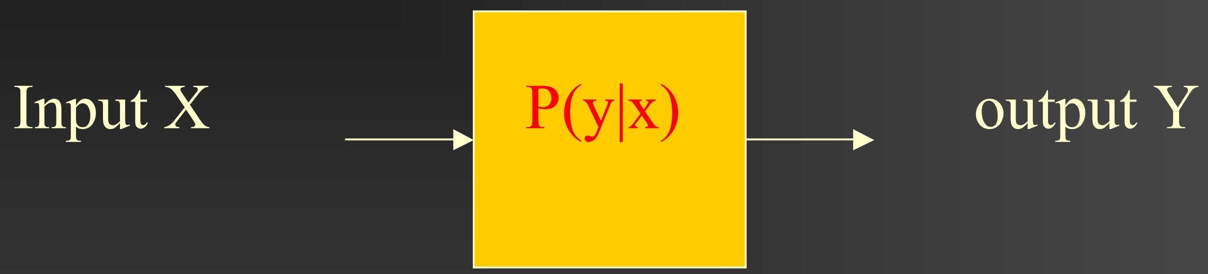
some channel models

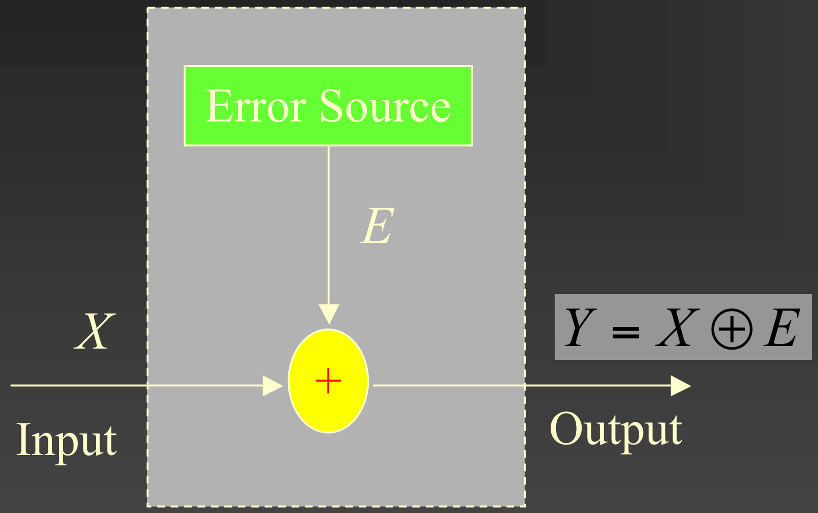


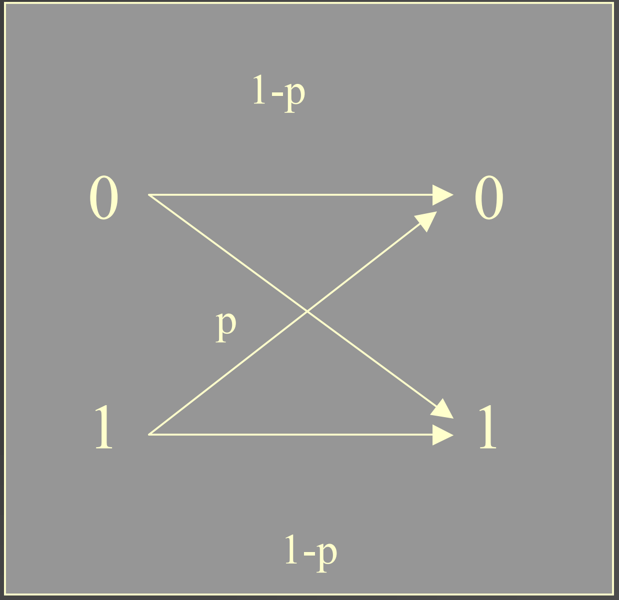
transition probabilities

memoryless:

- output at time i depends only on input at time i
- input and output alphabet finite

Example: binary symmetric channel (BSC)



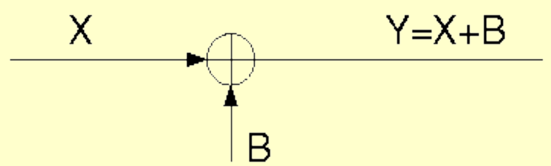


E is the binary error sequence s.t. P(1) = 1-P(0) = p

X is the binary information sequence

Y is the binary output sequence

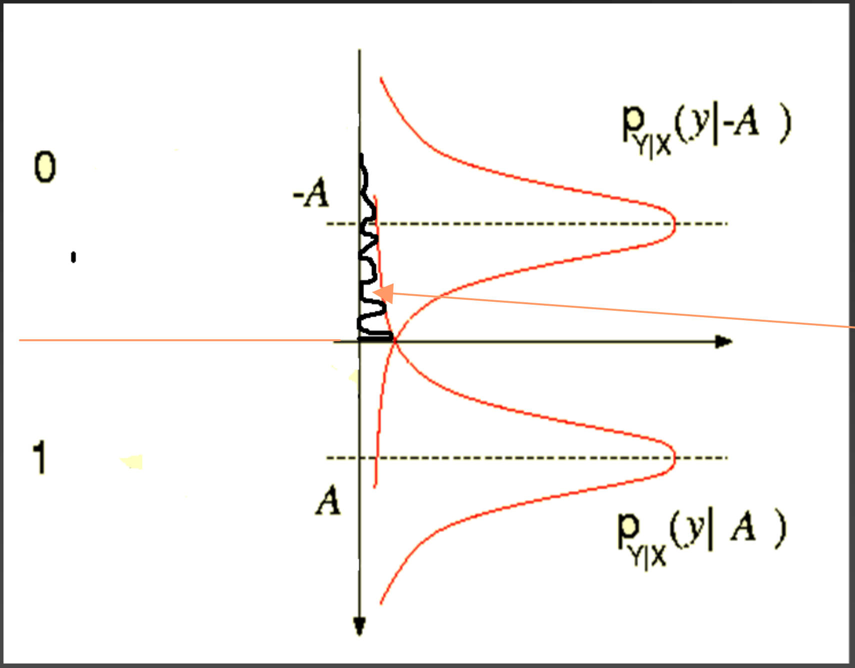
from AWGN to BSC

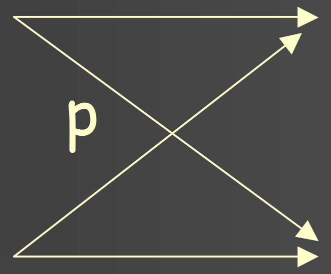


$X = \{x_1, x_2, \ldots, x_k\}$

$B \sim \mathcal{N}(0, \sigma^2)$

$Y \in \mathbb{R}$

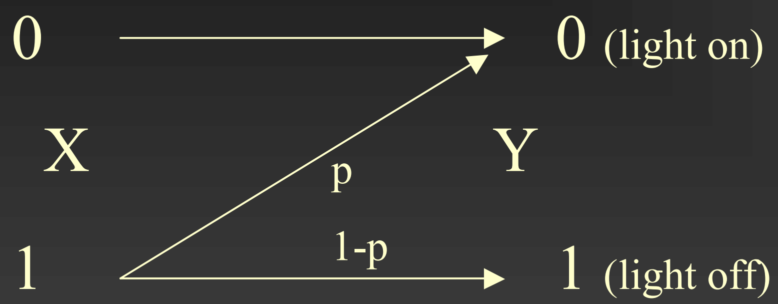




$$p_{Y|X}(y|X=x_k) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y-x_k)^2}{2\sigma^2}\right]$$

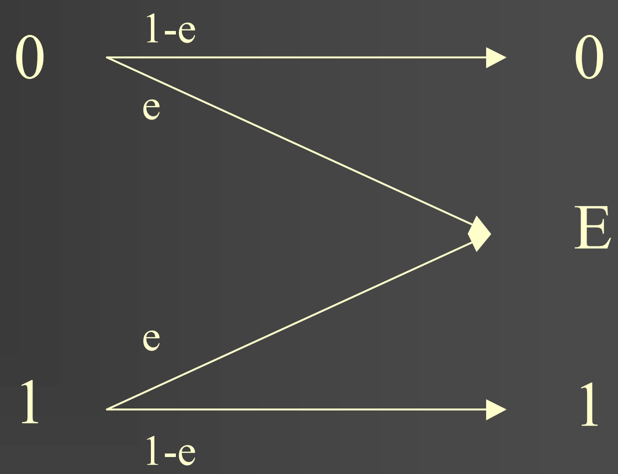
Homework: calculate the capacity as a function of A and σ^2

Other models



$P(X=0) = P_0$

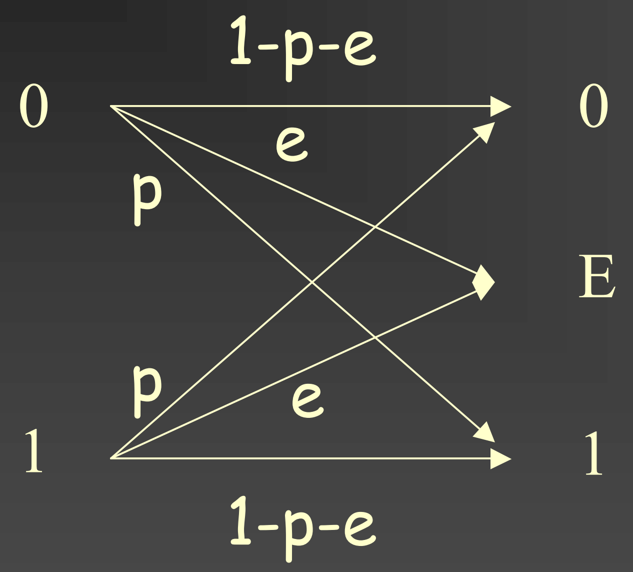
Z-channel (optical)
(MAC)



$P(X=0) = P_0$

Erasure channel

Erasure with errors



burst error model (Gilbert-Elliot)

Random error channel; outputs independent

Error Source → $P(0) = 1 - P(1);$

Burst error channel; outputs dependent

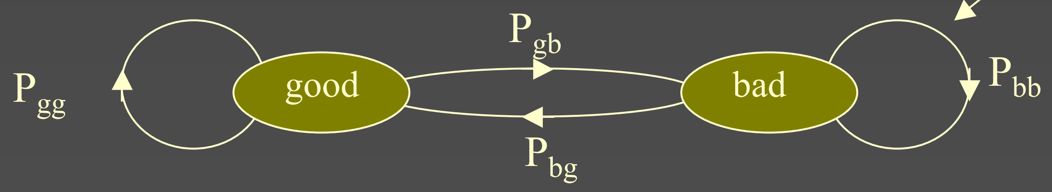
Error Source →

$P(0 \mid state = bad) = P(1 \mid state = bad) = 1/2;$

$P(0 \mid state = good) = 1 - P(1 \mid state = good) = 0.999$

State info: good or bad

transition probability



channel capacity:

I(X;Y) = H(X) - H(X|Y) = H(Y) – H(Y|X) (Shannon 1948)

$$H(X) \xrightarrow{X} \boxed{\text{channel}} \xrightarrow{Y} H(X|Y)$$

$$\max I(X;Y) = capacity$$

$$H = Entropy$$

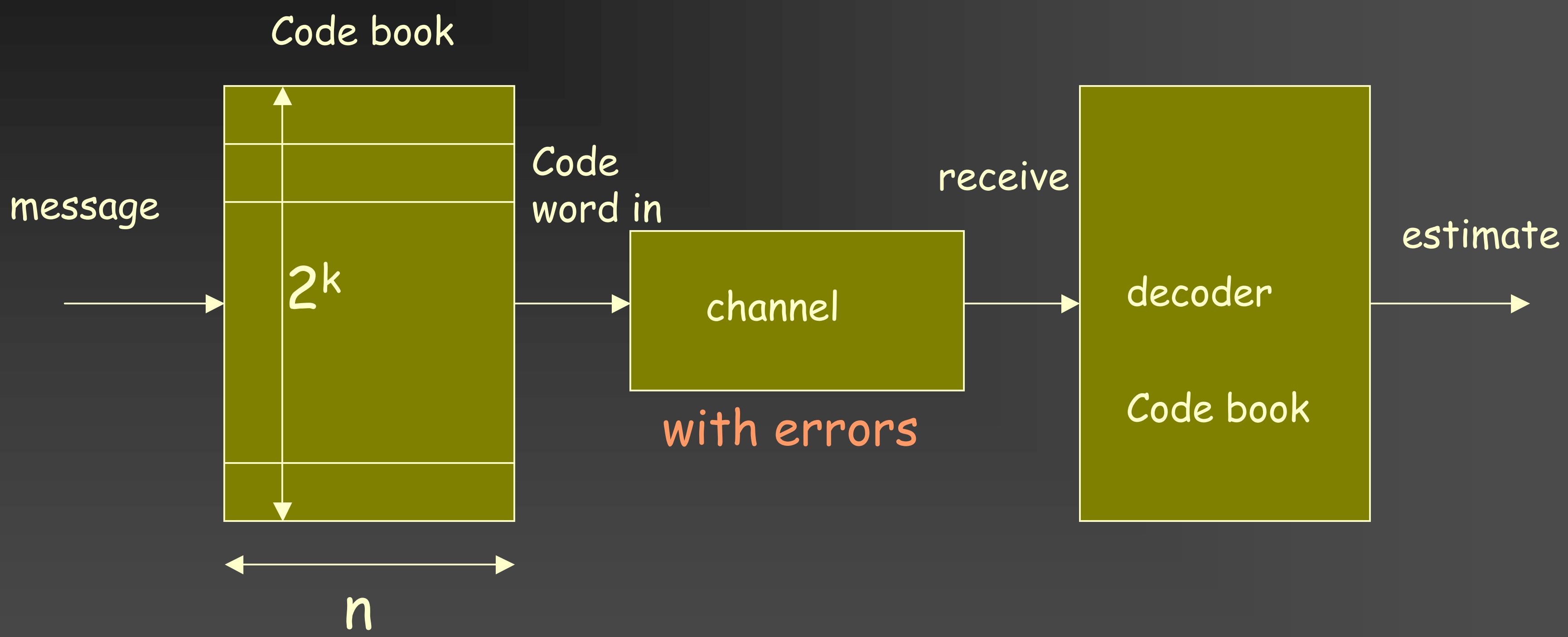
$$E(p) = \sum p(i) * I(i)$$

notes:

capacity depends on input probabilities

because the transition probabilites are fixed

Practical communication system design



There are 2^k code words of length n

k is the number of information bits transmitted in n channel uses

Channel capacity

Definition:

The rate R of a code is the ratio k/n, where

k is the number of information bits transmitted in n channel uses

Shannon showed that: :

for $R \le C$

encoding methods exist

with decoding error probability $\Rightarrow$ 0

Encoding and decoding according to Shannon

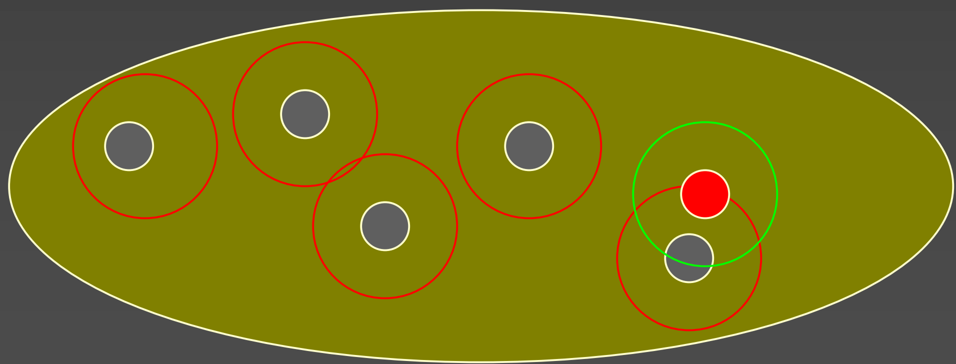
Code: 2^k binary codewords where $p(0) = P(1) = \frac{1}{2}$

Channel errors: $P(0 \rightarrow 1) = P(1 \rightarrow 0) = p$

i.e. # error sequences $\approx 2^{nh(p)}$

Decoder: search around received sequence for codeword

with $\approx np$ differences



space of 2^n binary sequences

decoding error probability

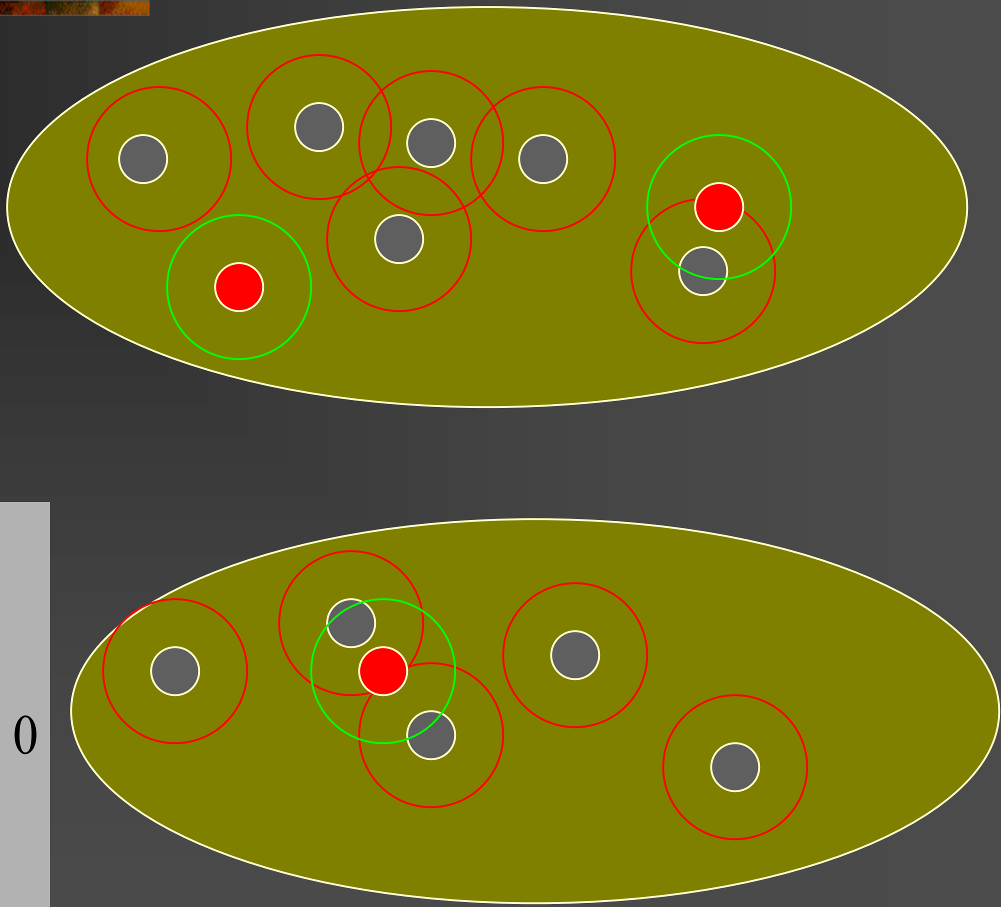
1. For t errors: $|t/n-p| > \epsilon$

   $\rightarrow$ 0 for $n \rightarrow \infty$

   (law of large numbers)

2. > 1 code word in region

   (codewords random)



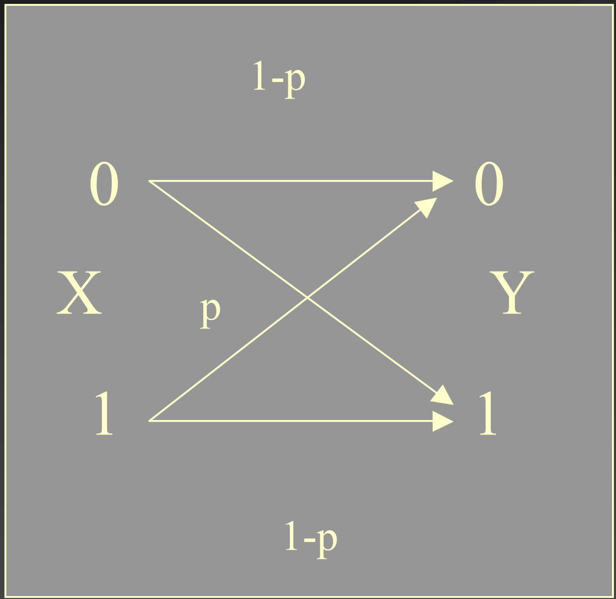
$$P(>1) \approx (2^k - 1)\frac{2^{nh(p)}}{2^n}$$

$$\rightarrow 2^{-n(1-h(p)-R)} = 2^{-n(C_{BSC}-R)} \rightarrow 0$$

for $\quad R = \frac{k}{n} < 1 - h(p)$

and $\quad n \rightarrow \infty$

channel capacity: the BSC



$I(X;Y) = H(Y) - H(Y|X)$

the maximum of $H(Y) = 1$

since Y is binary

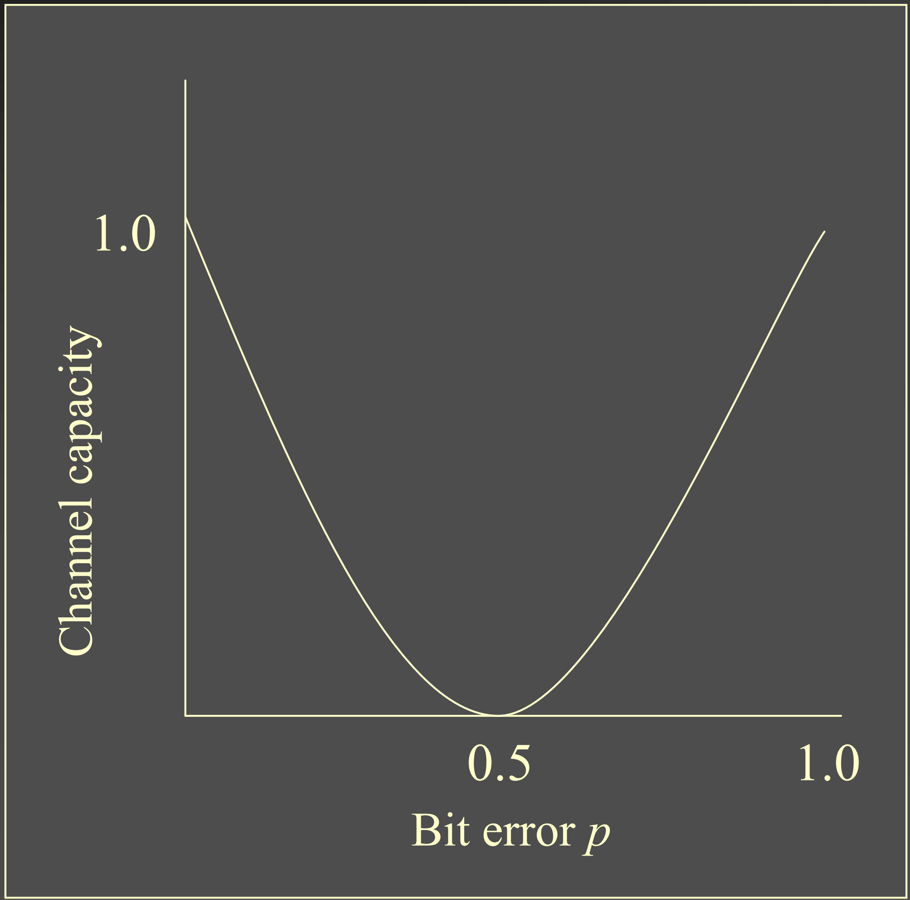
$H(Y|X) = h(p)$

$= P(X=0)h(p) + P(X=1)h(p)$

Conclusion: the capacity for the BSC $C_{BSC} = 1 - h(p)$

Homework: draw C_{BSC}, what happens for $p > \frac{1}{2}$

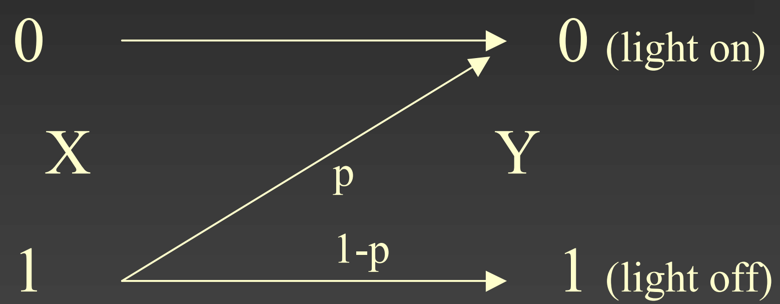
channel capacity: the BSC



Explain the behaviour!

channel capacity: the Z-channel

Application in optical communications



$P(X=0) = P_0$

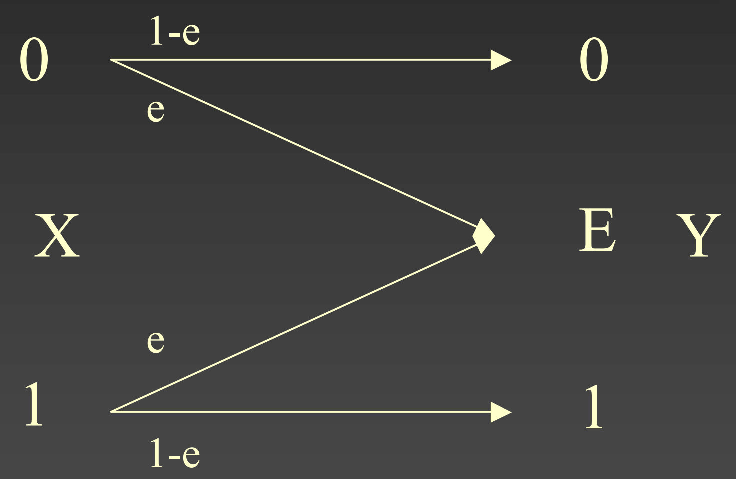
$H(Y) = h(P_0 + p(1 - P_0))$

$H(Y|X) = (1 - P_0)\, h(p)$

For capacity,
maximize $I(X;Y)$ over P_0

channel capacity: the erasure channel

Application: cdma detection



$$P(X=0) = P_0$$

$$I(X;Y) = H(Y) - H(Y|X)$$

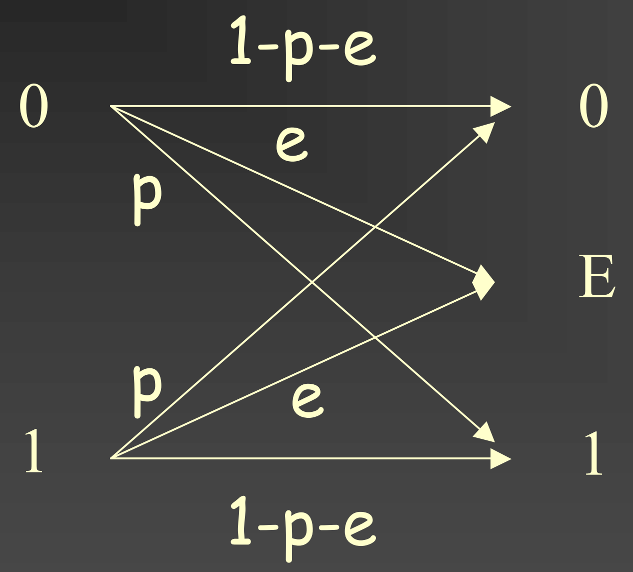
$$H(Y) = h(P_0)$$

$$H(Y|X) = e\, h(P_0)$$

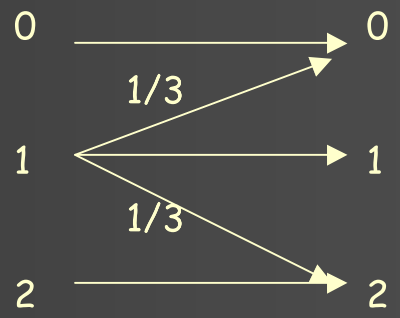
Thus $C_{erasure} = 1 - e$

(check!, draw and compare with BSC and Z)

Erasure with errors: calculate the capacity!



example



- Consider the following example

- For $P(0) = P(2) = p$, $P(1) = 1\text{-}2p$

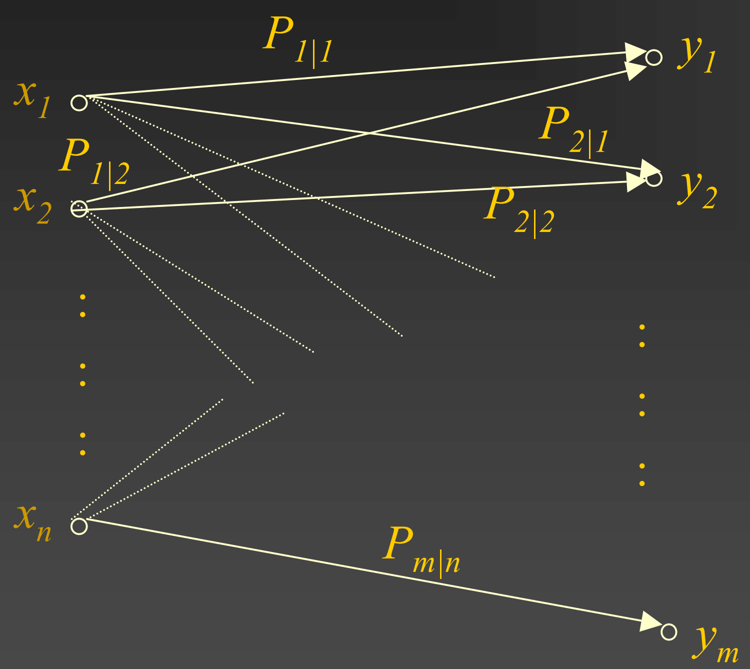
  $H(Y) = h(1/3 - 2p/3) + (2/3 + 2p/3)$; $H(Y|X) = (1\text{-}2p)\log_2 3$

  Q: maximize $H(Y) - H(Y|X)$ as a function of p

  Q: is this the capacity?

hint use the following: $\log_2 x = \ln x \,/\, \ln 2$; $d \ln x \,/\, dx = 1/x$

channel models: general diagram



Input alphabet $X = \{x_1, x_2, \ldots, x_n\}$

Output alphabet $Y = \{y_1, y_2, \ldots, y_m\}$

$P_{j|i} = P_{Y|X}(y_j|x_i)$

In general:

calculating capacity needs more theory

The statistical behavior of the channel is completely defined by the channel transition probabilities $P_{j|i} = P_{Y|X}(y_j|x_i)$

* clue:

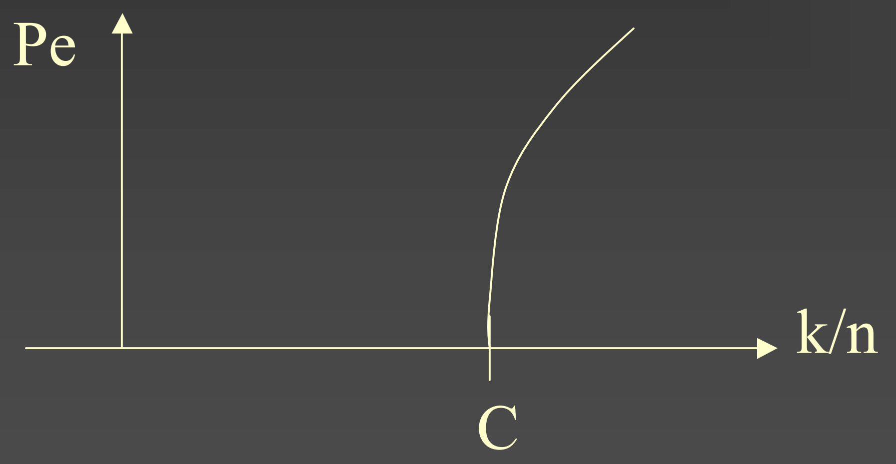
$I(X;Y)$

is convex ∩ in the input probabilities

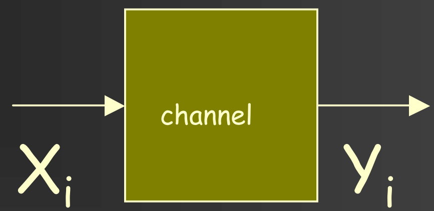
i.e. finding a maximum is simple

Channel capacity: converse

For R > C the decoding error probability > 0

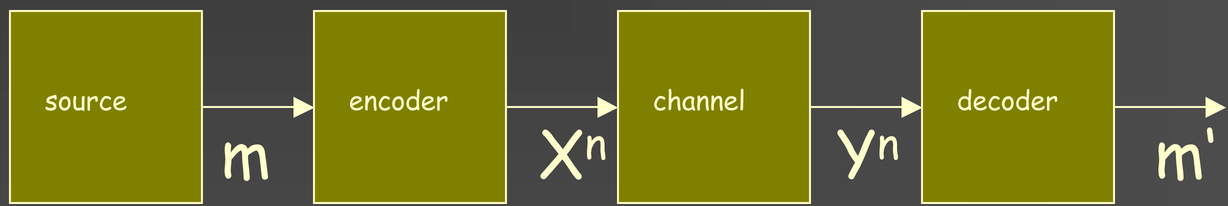


Converse: For a discrete memory less channel



$$I(X^n;Y^n) = H(Y^n) - \sum_{i=1}^{n} H(Y_i \mid X_i) \le \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i \mid X_i) = \sum_{i=1}^{n} I(X_i;Y_i) \le nC$$

Source generates one out of 2^k equiprobable messages



Let Pe = probability that $m' \neq m$

converse $R := k/n$

$$\left.\begin{aligned} k = H(M) &= I(M;Y^n) + H(M|Y^n) \\ &\le \underbrace{I(X^n;Y^n)}_{X^n \text{ is a function of } M} + \underbrace{1 + k\,Pe}_{\text{Fano}} \\ &\le nC + 1 + k\,Pe \end{aligned}\right\} \quad 1 - C\,n/k - 1/k \le Pe$$

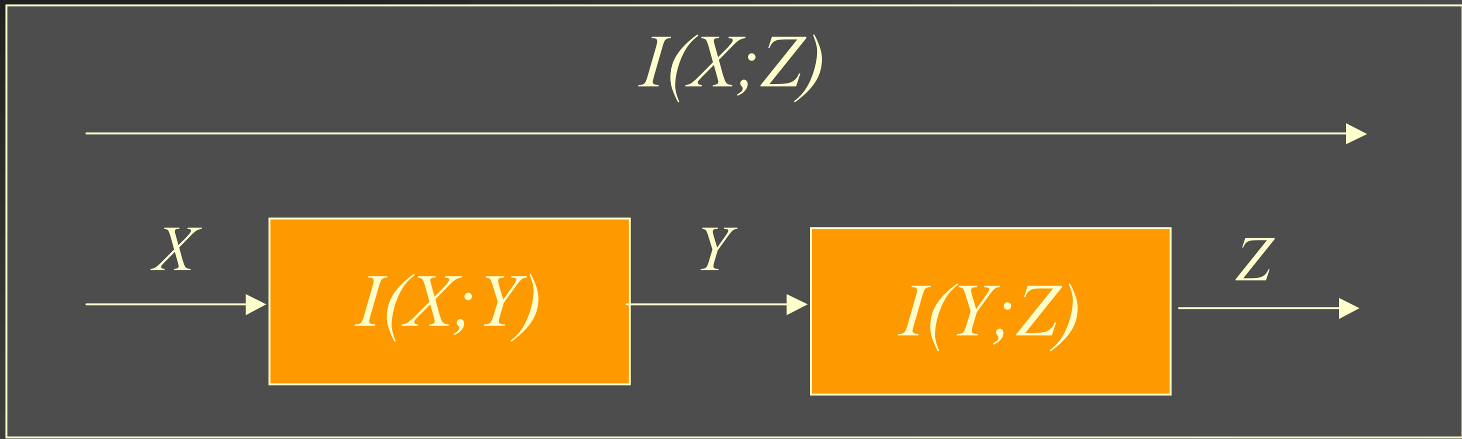
$Pe \ge 1 - C/R - 1/nR$

Hence: for large n, and $R > C$,

the probability of error $Pe > 0$

We used the data processing theorem

Cascading of Channels



The overall transmission rate I(X;Z) for the cascade can not be larger than I(Y;Z), that is:

$$I(X;Z) \leq I(Y;Z)$$