Logic and Proof

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Introduction to Logic

Logic concerns *statements* in some *language*.

The language can be natural (English, Latin, ...) or *formal*.

Some statements are *true*, others *false* or *meaningless*.

Logic concerns *relationships* between statements: consistency, entailment, ...

Logical *proofs* model human reasoning (supposedly).

Statements

Statements are declarative assertions:

Black is the colour of my true love's hair.

They are not greetings, questions or commands:

What is the colour of my true love's hair?

I wish my true love had hair.

Get a haircut!

Schematic Statements

Now let the *variables* $X, Y, Z, \ldots$ range over 'real' objects

Black is the colour of X*'s hair.*

Black is the colour of Y*.*

Z *is the colour of* Y*.*

Schematic statements can even express questions:

What things are black?

Interpretations and Validity

An *interpretation* maps meta-variables to real objects:

The interpretation $Y \mapsto$ coal *satisfies* the statement

Black is the colour of Y*.*

but the interpretation $Y \mapsto$ strawberries does not!

A statement A is *valid* if all interpretations satisfy A.

Consistency, or Satisfiability

A set S of statements is *consistent* if some interpretation satisfies all elements of S at the same time. Otherwise S is *inconsistent.*

Examples of inconsistent sets:

$$\{X \text{ part of } Y,\ Y \text{ part of } Z,\ X \text{ NOT part of } Z\}$$

$$\{n \text{ is a positive integer},\ n \neq 1,\ n \neq 2,\ \ldots\}$$

Satisfiable means the same as consistent.

Unsatisfiable means the same as inconsistent.

Entailment, or Logical Consequence

A set S of statements *entails* A if every interpretation that satisfies all elements of S, also satisfies A. We write $S \models A$.

$$\{X \text{ part of } Y,\ Y \text{ part of } Z\} \models X \text{ part of } Z$$

$$\{n \neq 1,\ n \neq 2,\ \ldots\} \models n \text{ is NOT a positive integer}$$

$S \models A$ if and only if $\{\neg A\} \cup S$ is inconsistent

$\models A$ if and only if A is valid, if and only if $\{\neg A\}$ is inconsistent.

Inference

We want to check A is valid.

Checking all interpretations can be effective — but what if there are infinitely many?

Let $\{A_1, \ldots, A_n\} \models B$. If $A_1, \ldots, A_n$ are true then B must be true. Write this as the *inference rule*

$$\frac{A_1 \quad \ldots \quad A_n}{B}$$

We can use inference rules to construct finite proofs!

Schematic Inference Rules

$$\frac{X \text{ part of } Y \qquad Y \text{ part of } Z}{X \text{ part of } Z}$$

A valid inference:

$$\frac{\text{spoke part of wheel} \qquad \text{wheel part of bike}}{\text{spoke part of bike}}$$

An inference may be valid even if the premises are false!

$$\frac{\text{cow part of chair} \qquad \text{chair part of ant}}{\text{cow part of ant}}$$

Survey of Formal Logics

propositional logic is traditional *boolean algebra*.

first-order logic can say *for all* and *there exists*.

higher-order logic reasons about sets and functions.

modal/temporal logics reason about what *must*, or *may*, happen.

type theories support *constructive* mathematics.

All have been used to prove correctness of computer systems.

Why Should the Language be Formal?

Consider this ‘definition’: (Berry’s paradox)

> The smallest positive integer not definable using nine words

Greater than *The number of atoms in the Milky Way galaxy*

This number is so large, it is greater than *itself*!

- A formal language prevents AMBIGUITY.

Syntax of Propositional Logic

$P, Q, R, \ldots$	propositional letter
t	true
f	false
$\neg A$	not A
$A \wedge B$	A and B
$A \vee B$	A or B
$A \to B$	if A then B
$A \leftrightarrow B$	A if and only if B

Semantics of Propositional Logic

$\neg$, $\land$, $\lor$, $\to$ and $\leftrightarrow$ are *truth-functional*: functions of their operands.

A	B	$\neg A$	$A \land B$	$A \lor B$	$A \to B$	$A \leftrightarrow B$
t	**t**	**f**	**t**	**t**	**t**	**t**
t	**f**	**f**	**f**	**t**	**f**	**f**
f	**t**	**t**	**f**	**t**	**t**	**f**
f	**f**	**t**	**f**	**f**	**t**	**t**

Interpretations of Propositional Logic

An *interpretation* is a function from the propositional letters to $\{\mathbf{t}, \mathbf{f}\}$.

Interpretation I *satisfies* a formula A if the formula evaluates to **t**.

Write $\models_I A$

A is *valid* (a *tautology*) if every interpretation satisfies A.

Write $\models A$

S is *satisfiable* if some interpretation satisfies every formula in S.

Implication, Entailment, Equivalence

$A \to B$ means simply $\neg A \lor B$.

$A \models B$ means if $\models_I A$ then $\models_I B$ for every interpretation I.

$A \models B$ if and only if $\models A \to B$.

Equivalence

$A \simeq B$ means $A \models B$ and $B \models A$.

$A \simeq B$ if and only if $\models A \leftrightarrow B$.

Equivalences

$$A \land A \simeq A$$
$$A \land B \simeq B \land A$$
$$(A \land B) \land C \simeq A \land (B \land C)$$
$$A \lor (B \land C) \simeq (A \lor B) \land (A \lor C)$$
$$A \land \mathbf{f} \simeq \mathbf{f}$$
$$A \land \mathbf{t} \simeq A$$
$$A \land \neg A \simeq \mathbf{f}$$

Dual versions: exchange $\land$ with $\lor$ and **t** with **f** in any equivalence

Negation Normal Form

1. Get rid of $\leftrightarrow$ and $\to$, leaving just $\land$, $\lor$, $\neg$:

$$A \leftrightarrow B \simeq (A \to B) \land (B \to A)$$
$$A \to B \simeq \neg A \lor B$$

2. Push negations in, using de Morgan's laws:

$$\neg\neg A \simeq A$$
$$\neg(A \land B) \simeq \neg A \lor \neg B$$
$$\neg(A \lor B) \simeq \neg A \land \neg B$$

From NNF to Conjunctive Normal Form

3. Push disjunctions in, using distributive laws:

$$A \lor (B \land C) \simeq (A \lor B) \land (A \lor C)$$
$$(B \land C) \lor A \simeq (B \lor A) \land (C \lor A)$$

4. Simplify:

- Delete any disjunction containing P and $\neg P$
- Delete any disjunction that includes another: for example, in $(P \lor Q) \land P$, delete $P \lor Q$.
- Replace $(P \lor A) \land (\neg P \lor A)$ by A

Converting a Non-Tautology to CNF

$$P \vee Q \rightarrow Q \vee R$$

1. Elim $\rightarrow$: $\neg(P \vee Q) \vee (Q \vee R)$
2. Push $\neg$ in: $(\neg P \wedge \neg Q) \vee (Q \vee R)$
3. Push $\vee$ in: $(\neg P \vee Q \vee R) \wedge (\neg Q \vee Q \vee R)$
4. Simplify: $\neg P \vee Q \vee R$

Not a tautology: try $P \mapsto$ **t**, $Q \mapsto$ **f**, $R \mapsto$ **f**

Tautology checking using CNF

$$((P \rightarrow Q) \rightarrow P) \rightarrow P$$

1. Elim $\rightarrow$: $\neg[\neg(\neg P \vee Q) \vee P] \vee P$
2. Push $\neg$ in: $[\neg\neg(\neg P \vee Q) \wedge \neg P] \vee P$
   $[(\neg P \vee Q) \wedge \neg P] \vee P$
3. Push $\vee$ in: $(\neg P \vee Q \vee P) \wedge (\neg P \vee P)$
4. Simplify: **t** $\wedge$ **t**
   t *It's a tautology!*

A Simple Proof System

Axiom Schemes

K $\quad A \rightarrow (B \rightarrow A)$

S $\quad (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$

DN $\quad \neg\neg A \rightarrow A$

Inference Rule: Modus Ponens

$$\frac{A \rightarrow B \qquad A}{B}$$

A Simple (?) Proof of $A \rightarrow A$

$(A \rightarrow ((D \rightarrow A) \rightarrow A)) \rightarrow$		(1)
$((A \rightarrow (D \rightarrow A)) \rightarrow (A \rightarrow A))$	by S	
$A \rightarrow ((D \rightarrow A) \rightarrow A)$	by K	(2)
$(A \rightarrow (D \rightarrow A)) \rightarrow (A \rightarrow A)$	by MP, (1), (2)	(3)
$A \rightarrow (D \rightarrow A)$	by K	(4)
$A \rightarrow A$	by MP, (3), (4)	(5)

Some Facts about Deducibility

A is *deducible from* the set S if there is a finite proof of A starting from elements of S. Write $S \vdash A$.

Soundness Theorem. If $S \vdash A$ then $S \models A$.

Completeness Theorem. If $S \models A$ then $S \vdash A$.

Deduction Theorem. If $S \cup \{A\} \vdash B$ then $S \vdash A \rightarrow B$.

Gentzen's Natural Deduction Systems

The context of *assumptions* may vary.

Each logical connective is defined *independently*.

The *introduction* rule for $\wedge$ shows how to deduce $A \wedge B$:

$$\frac{A \qquad B}{A \wedge B}$$

The *elimination* rules for $\wedge$ shows what to deduce *from* $A \wedge B$:

$$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

The Sequent Calculus

Sequent $A_1, \dots, A_m \Rightarrow B_1, \dots, B_n$ means,

if $A_1 \land \dots \land A_m$ then $B_1 \lor \dots \lor B_n$

$A_1, \dots, A_m$ are *assumptions*; $B_1, \dots, B_n$ are *goals*

Γ and Δ are *sets* in $\Gamma \Rightarrow \Delta$

The sequent $A, \Gamma \Rightarrow A, \Delta$ is trivially true (*basic sequent*).

Sequent Calculus Rules

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (cut)$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ (\neg l) \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}\ (\neg r)$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \land B, \Gamma \Rightarrow \Delta}\ (\land l) \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \land B}\ (\land r)$$

More Sequent Calculus Rules

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \lor B, \Gamma \Rightarrow \Delta}\ (\lor l) \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \lor B}\ (\lor r)$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ (\to l) \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ (\to r)$$

Easy Sequent Calculus Proofs

$$\dfrac{\dfrac{\overline{A, B \Rightarrow A}}{A \land B \Rightarrow A}\ (\land l)}{\Rightarrow (A \land B) \to A}\ (\to r)$$

$$\dfrac{\dfrac{\dfrac{\overline{A, B \Rightarrow B, A}}{A \Rightarrow B, B \to A}\ (\to r)}{\Rightarrow A \to B,\ B \to A}\ (\to r)}{\Rightarrow (A \to B) \lor (B \to A)}\ (\lor r)$$

Part of a Distributive Law

$$\dfrac{\dfrac{\dfrac{\overline{A \Rightarrow A, B} \qquad \dfrac{\overline{B, C \Rightarrow A, B}}{B \land C \Rightarrow A, B}\ (\land l)}{A \lor (B \land C) \Rightarrow A, B}\ (\lor l)}{A \lor (B \land C) \Rightarrow A \lor B}\ (\lor r) \qquad \text{similar}}{A \lor (B \land C) \Rightarrow (A \lor B) \land (A \lor C)}\ (\land r)$$

Second subtree proves $A \lor (B \land C) \Rightarrow A \lor C$ similarly

A Failed Proof

$$\dfrac{\dfrac{\dfrac{A \Rightarrow B, C \qquad \overline{B \Rightarrow B, C}}{A \lor B \Rightarrow B, C}\ (\lor l)}{A \lor B \Rightarrow B \lor C}\ (\lor r)}{\Rightarrow (A \lor B) \to (B \lor C)}\ (\to r)$$

$A \mapsto \mathbf{t}$, $B \mapsto \mathbf{f}$, $C \mapsto \mathbf{f}$ falsifies unproved sequent!

Outline of First-Order Logic

Reasons about *functions* and *relations* over a set of *individuals*:

$$\frac{\text{father}(\text{father}(x)) = \text{father}(\text{father}(y))}{\text{cousin}(x, y)}$$

Reasons about *all* and *some* individuals:

$$\frac{\text{All men are mortal} \qquad \text{Socrates is a man}}{\text{Socrates is mortal}}$$

Cannot reason about *all functions* or *all relations*, etc.

Function Symbols; Terms

Each *function symbol* stands for an n-place function.

A *constant symbol* is a 0-place function symbol.

A *variable* ranges over all individuals.

A *term* is a variable, constant or a function application

$$f(t_1, \ldots, t_n)$$

where f is an n-place function symbol and t_1, ..., t_n are terms.

We choose the language, adopting any desired function symbols.

Relation Symbols; Formulae

Each *relation symbol* stands for an n-place relation.

Equality is the 2-place relation symbol $=$

An *atomic formula* has the form $R(t_1, \ldots, t_n)$ where R is an n-place relation symbol and t_1, ..., t_n are terms.

A *formula* is built up from atomic formulæ using $\neg$, $\wedge$, $\vee$, and so forth.

Later, we can add *quantifiers*.

The Power of Quantifier-Free FOL

It is surprisingly expressive, if we include strong induction rules.

We can easily prove the equivalence of mathematical functions:

$$\begin{array}{ll} p(z,0) = 1 & q(z,1) = z \\ p(z,n+1) = p(z,n) \times z & q(z, 2 \times n) = q(z \times z, n) \\ & q(z, 2 \times n + 1) = q(z \times z, n) \times z \end{array}$$

The prover ACL2 uses this logic to do major hardware proofs.

Universal and Existential Quantifiers

$\forall x\, A$ for all x, the formula A holds

$\exists x\, A$ there exists x such that A holds

Syntactic variations:

$\forall xyz\, A$ abbreviates $\forall x\, \forall y\, \forall z\, A$

$\forall z\,.\, A \wedge B$ is an alternative to $\forall z\, (A \wedge B)$

The variable x is *bound* in $\forall x\, A$; compare with $\int f(x)dx$

The Expressiveness of Quantifiers

All men are mortal:

$$\forall x\, (\text{man}(x) \to \text{mortal}(x))$$

All mothers are female:

$$\forall x\, \text{female}(\text{mother}(x))$$

There exists a unique x *such that* A, sometimes written $\exists!x\, A$

$$\exists x\, [A(x) \wedge \forall y\, (A(y) \to y = x)]$$

The Point of Semantics

We have to attach meanings to symbols like 1, $+$, $<$, etc.

Why is this necessary? Why can't 1 just mean 1??

The point is that mathematics derives its flexibility from allowing different interpretations of symbols.

- A *group* has a unit 1, a product $x \cdot y$ and inverse x^{-1}.
- In the most important uses of groups, 1 isn't a number but a 'unit permutation', 'unit rotation', etc.

Constants: Interpreting $\mathrm{mortal}(\mathrm{Socrates})$

An interpretation $\mathcal{I} = (D, I)$ defines the *semantics* of a first-order language.

D is a non-empty set, called the *domain* or *universe*.

I maps symbols to 'real' elements, functions and relations:

c a *constant* symbol	$I[c] \in D$
f an n-place *function* symbol	$I[f] \in D^n \to D$
P an n-place *relation* symbol	$I[P] \in D^n \to \{\mathbf{t}, \mathbf{f}\}$

Variables: Interpreting $\mathrm{cousin}(\mathrm{Charles}, y)$

A *valuation* V : variables $\to D$ supplies the values of free variables.

An interpretation $\mathcal{I}$ and valuation function V jointly specify the value of any term t by the obvious recursion.

This value is written $\mathcal{I}_V[t]$, and here are the recursion rules:

$$\mathcal{I}_V[x] \stackrel{\text{def}}{=} V(x) \qquad \text{if } x \text{ is a variable}$$

$$\mathcal{I}_V[c] \stackrel{\text{def}}{=} I[c]$$

$$\mathcal{I}_V[f(t_1, \ldots, t_n)] \stackrel{\text{def}}{=} I[f](\mathcal{I}_V[t_1], \ldots, \mathcal{I}_V[t_n])$$

Tarski's Truth-Definition

An interpretation $\mathcal{I}$ and valuation function V similarly specify the truth value (**t** or **f**) of any formula A.

Quantifiers are the only problem, as they bind variables.

$V\{a/x\}$ is the valuation that maps x to a and is otherwise like V.

With the help of $V\{a/x\}$, we now formally define $\models_{\mathcal{I},V} A$, the truth value of A.

The Meaning of Truth—In FOL!

For interpretation $\mathcal{I}$ and valuation V, define $\models_{\mathcal{I},V}$ by recursion.

$\models_{\mathcal{I},V} P(t)$	if $\mathcal{I}_V[t] \in I[P]$ holds
$\models_{\mathcal{I},V} t = u$	if $\mathcal{I}_V[t]$ equals $\mathcal{I}_V[u]$
$\models_{\mathcal{I},V} A \land B$	if $\models_{\mathcal{I},V} A$ and $\models_{\mathcal{I},V} B$
$\models_{\mathcal{I},V} \exists x\, A$	if $\models_{\mathcal{I},V\{m/x\}} A$ holds for some $m \in D$

Finally, we define

$\models_{\mathcal{I}} A$	if $\models_{\mathcal{I},V} A$ holds for all V.

A *closed* formula A is *satisfiable* if $\models_{\mathcal{I}} A$ for some $\mathcal{I}$.

Free vs Bound Variables

All occurrences of x in $\forall x\, A$ and $\exists x\, A$ are *bound*

An occurrence of x is *free* if it is not bound:

$$\forall y\, \exists z\, R(y, z, f(y, x))$$

In this formula, y and z are bound while x is free.

We may *rename* bound variables without affecting the meaning:

$$\forall w\, \exists z'\, R(w, z', f(w, x))$$

Substitution for Free Variables

$A[t/x]$ means *substitute* t *for* x *in* A:

$$\begin{array}{rcl} (B \wedge C)[t/x] & \textit{is} & B[t/x] \wedge C[t/x] \\ (\forall x\, B)[t/x] & \textit{is} & \forall x\, B \\ (\forall y\, B)[t/x] & \textit{is} & \forall y\, B[t/x] \qquad (x \neq y) \\ (P(u))[t/x] & \textit{is} & P(u[t/x]) \end{array}$$

When substituting $A[t/x]$, no variable of t may be bound in A!

Example: $(\forall y\ (x = y))\,[y/x]$ IS NOT EQUIVALENT TO $\forall y\ (y = y)$

Some Equivalences for Quantifiers

$$\neg(\forall x\, A) \simeq \exists x\, \neg A$$
$$\forall x\, A \simeq \forall x\, A \wedge A[t/x]$$
$$(\forall x\, A) \wedge (\forall x\, B) \simeq \forall x\, (A \wedge B)$$

BUT WE DO NOT HAVE $(\forall x\, A) \vee (\forall x\, B) \simeq \forall x\, (A \vee B)$.

Dual versions: exchange $\forall$ with $\exists$ and $\wedge$ with $\vee$

Further Quantifier Equivalences

These hold only if x is not free in B.

$$(\forall x\, A) \wedge B \simeq \forall x\, (A \wedge B)$$
$$(\forall x\, A) \vee B \simeq \forall x\, (A \vee B)$$
$$(\forall x\, A) \to B \simeq \exists x\, (A \to B)$$

These let us expand or contract a quantifier's scope.

Reasoning by Equivalences

$$\begin{aligned} \exists x\, (x = a \wedge P(x)) &\simeq \exists x\, (x = a \wedge P(a)) \\ &\simeq \exists x\, (x = a) \wedge P(a) \\ &\simeq P(a) \end{aligned}$$

$$\begin{aligned} &\exists z\, (P(z) \to P(a) \wedge P(b)) \\ &\quad \simeq \forall z\, P(z) \to P(a) \wedge P(b) \\ &\quad \simeq \forall z\, P(z) \wedge P(a) \wedge P(b) \to P(a) \wedge P(b) \\ &\quad \simeq \mathbf{t} \end{aligned}$$

Sequent Calculus Rules for $\forall$

$$\frac{A[t/x], \Gamma \Rightarrow \Delta}{\forall x\, A, \Gamma \Rightarrow \Delta}\ (\forall l) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \forall x\, A}\ (\forall r)$$

Rule $(\forall l)$ can create many instances of $\forall x\, A$

Rule $(\forall r)$ holds *provided* x is not free in the conclusion!

NOT allowed to prove

$$\frac{\overline{P(y) \Rightarrow P(y)}}{P(y) \Rightarrow \forall y\, P(y)}\ (\forall r)$$

THIS IS NONSENSE!

A Simple Example of the $\forall$ Rules

$$\cfrac{\cfrac{\overline{P(f(y)) \Rightarrow P(f(y))}}{\forall x\, P(x) \Rightarrow P(f(y))}\ (\forall l)}{\forall x\, P(x) \Rightarrow \forall y\, P(f(y))}\ (\forall r)$$

A Not-So-Simple Example of the $\forall$ Rules

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{P \Rightarrow Q(y), P} \qquad \overline{P, Q(y) \Rightarrow Q(y)}}{P,\ P \to Q(y) \Rightarrow Q(y)}\,(\to l)}{P,\ \forall x\,(P \to Q(x)) \Rightarrow Q(y)}\,(\forall l)}{P,\ \forall x\,(P \to Q(x)) \Rightarrow \forall y\,Q(y)}\,(\forall r)}{\forall x\,(P \to Q(x)) \Rightarrow P \to \forall y\,Q(y)}\,(\to r)}{}
$$

In $(\forall l)$, we must replace x by y.

Sequent Calculus Rules for $\exists$

$$
\dfrac{A, \Gamma \Rightarrow \Delta}{\exists x\, A, \Gamma \Rightarrow \Delta}\,(\exists l) \qquad \dfrac{\Gamma \Rightarrow \Delta, A[t/x]}{\Gamma \Rightarrow \Delta, \exists x\, A}\,(\exists r)
$$

Rule $(\exists l)$ holds *provided* x is not free in the conclusion!

Rule $(\exists r)$ can create many instances of $\exists x\, A$

For example, to prove this counter-intuitive formula:

$$\exists z\,(P(z) \to P(a) \land P(b))$$

Part of the $\exists$ Distributive Law

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{P(x) \Rightarrow P(x), Q(x)}}{P(x) \Rightarrow P(x) \lor Q(x)}\,(\lor r)}{P(x) \Rightarrow \exists y\,(P(y) \lor Q(y))}\,(\exists r)}{\exists x\, P(x) \Rightarrow \exists y\,(P(y) \lor Q(y))}\,(\exists l) \qquad \dfrac{\textit{similar}}{\exists x\, Q(x) \Rightarrow \exists y\ \ldots}\,(\exists l)}{\exists x\, P(x) \lor \exists x\, Q(x) \Rightarrow \exists y\,(P(y) \lor Q(y))}\,(\lor l)
$$

Second subtree proves $\exists x\, Q(x) \Rightarrow \exists y\,(P(y) \lor Q(y))$ similarly

In $(\exists r)$, we must replace y by x.

A Failed Proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(x), Q(y) \Rightarrow P(x) \land Q(x)}{P(x), Q(y) \Rightarrow \exists z\,(P(z) \land Q(z))}\,(\exists r)}{P(x), \exists x\, Q(x) \Rightarrow \exists z\,(P(z) \land Q(z))}\,(\exists l)}{\exists x\, P(x), \exists x\, Q(x) \Rightarrow \exists z\,(P(z) \land Q(z))}\,(\exists l)}{\exists x\, P(x) \land \exists x\, Q(x) \Rightarrow \exists z\,(P(z) \land Q(z))}\,(\land l)
$$

We cannot use $(\exists l)$ twice with the same variable

This attempt renames the x in $\exists x\, Q(x)$, to get $\exists y\, Q(y)$

Clause Form

Clause: a disjunction of *literals*

$$\neg K_1 \lor \cdots \lor \neg K_m \lor L_1 \lor \cdots \lor L_n$$

Set notation: $\{\neg K_1, \ldots, \neg K_m, L_1, \ldots, L_n\}$

Kowalski notation: $K_1, \cdots, K_m \to L_1, \cdots, L_n$

$L_1, \cdots, L_n \leftarrow K_1, \cdots, K_m$

Empty clause: $\{\}$ or $\square$

Empty clause is equivalent to **f**, meaning CONTRADICTION!

Outline of Clause Form Methods

To prove A, obtain a contradiction from $\neg A$:

1. Translate $\neg A$ into CNF as $A_1 \land \cdots \land A_m$
2. This is the set of clauses $A_1, \ldots, A_m$
3. Transform the clause set, *preserving consistency*

Deducing the *empty clause* refutes $\neg A$.

An empty *clause set* (all clauses deleted) means $\neg A$ is satisfiable.

The basis for SAT SOLVERS and RESOLUTION PROVERS.

The Davis-Putnam-Logeman-Loveland Method

1. Delete tautological clauses: $\{P, \neg P, \dots\}$
2. For each unit clause $\{L\}$,
   - delete all clauses containing L
   - delete $\neg L$ from all clauses
3. Delete all clauses containing *pure literals*
4. Perform a *case split* on some literal; STOP if a model is found

DPLL is a **decision procedure**: it finds a contradiction or a model.

Davis-Putnam on a Non-Tautology

Consider $P \vee Q \to Q \vee R$

Clauses are $\{P, Q\}$ $\{\neg Q\}$ $\{\neg R\}$

$\{P, Q\}$	$\{\neg Q\}$	$\{\neg R\}$	initial clauses
$\{P\}$		$\{\neg R\}$	unit $\neg Q$
		$\{\neg R\}$	unit P (also pure)
			unit $\neg R$ (also pure)

ALL CLAUSES DELETED! Clauses satisfiable by $P \mapsto \mathbf{t}$, $Q \mapsto \mathbf{f}$, $R \mapsto \mathbf{f}$

Example of a Case Split on P

$\{\neg Q, R\}$ $\{\neg R, P\}$ $\{\neg R, Q\}$ $\{\neg P, Q, R\}$ $\{P, Q\}$ $\{\neg P, \neg Q\}$

$\{\neg Q, R\}$	$\{\neg R, Q\}$	$\{Q, R\}$	$\{\neg Q\}$	if P is true
	$\{\neg R\}$	$\{R\}$		unit $\neg Q$
	$\{\}$			unit R
$\{\neg Q, R\}$	$\{\neg R\}$	$\{\neg R, Q\}$	$\{Q\}$	if P is false
$\{\neg Q\}$			$\{Q\}$	unit $\neg R$
			$\{\}$	unit $\neg Q$

Both cases yield contradictions: the clauses are INCONSISTENT!

SAT solvers in the Real World

- Progressed from joke to killer technology in 10 years.
- Princeton's zChaff has solved problems with more than one million variables and 10 million clauses.
- Applications include finding bugs in device drivers (Microsoft's SLAM project).
- Typical approach: approximate the problem with a finite model; encode it using Boolean logic; supply to a SAT solver.

The Resolution Rule

From $B \vee A$ and $\neg B \vee C$ infer $A \vee C$

In set notation,

$$\frac{\{B, A_1, \dots, A_m\} \quad \{\neg B, C_1, \dots, C_n\}}{\{A_1, \dots, A_m, C_1, \dots, C_n\}}$$

Some special cases: *(remember that* $\square$ *is just* $\{\}$*)*

$$\frac{\{B\} \quad \{\neg B, C_1, \dots, C_n\}}{\{C_1, \dots, C_n\}} \qquad \frac{\{B\} \quad \{\neg B\}}{\square}$$

Simple Example: Proving $P \wedge Q \to Q \wedge P$

Hint: use $\neg(A \to B) \simeq A \wedge \neg B$

1. Negate! $\neg[P \wedge Q \to Q \wedge P]$
2. Push $\neg$ in: $(P \wedge Q) \wedge \neg(Q \wedge P)$
   $(P \wedge Q) \wedge (\neg Q \vee \neg P)$

Clauses: $\{P\}$ $\{Q\}$ $\{\neg Q, \neg P\}$

Resolve $\{P\}$ and $\{\neg Q, \neg P\}$ getting $\{\neg Q\}$.

Resolve $\{Q\}$ and $\{\neg Q\}$ getting $\square$: we have refuted the negation.

Another Example

Refute $\neg[(P \vee Q) \wedge (P \vee R) \to P \vee (Q \wedge R)]$

From $(P \vee Q) \wedge (P \vee R)$, get clauses $\{P, Q\}$ and $\{P, R\}$.

From $\neg [P \vee (Q \wedge R)]$ get clauses $\{\neg P\}$ and $\{\neg Q, \neg R\}$.

Resolve $\{\neg P\}$ and $\{P, Q\}$ getting $\{Q\}$.

Resolve $\{\neg P\}$ and $\{P, R\}$ getting $\{R\}$.

Resolve $\{Q\}$ and $\{\neg Q, \neg R\}$ getting $\{\neg R\}$.

Resolve $\{R\}$ and $\{\neg R\}$ getting $\square$, contradiction.

The Saturation Algorithm

At start, all clauses are *passive*. None are *active*.

1. Transfer a clause (*current*) from *passive* to *active*.
2. Form all resolvents between *current* and an *active* clause.
3. Use new clauses to simplify both *passive* and *active*.
4. Put the new clauses into *passive*.

Repeat until CONTRADICTION found or *passive* becomes empty.

Heuristics and Hacks for Resolution

Orderings to focus the search on specific literals

Subsumption, or deleting redundant clauses

Indexing: elaborate data structures for speed

Preprocessing: removing tautologies, symmetries . . .

Weighting: giving priority to "good" clauses over those containing unwanted constants

Reducing FOL to Propositional Logic

Prenex:	Move quantifiers to the front (JUST FOR NOW!)
Skolemize:	Remove quantifiers, preserving *consistency*
Herbrand models:	Reduce the class of interpretations
Herbrand's Thm:	Contradictions have *finite*, *ground* proofs
Unification:	Automatically find the right instantiations

Finally, combine unification with *resolution*

Prenex Normal Form

Convert to Negation Normal Form using additionally

$$\neg(\forall x\, A) \simeq \exists x\, \neg A$$
$$\neg(\exists x\, A) \simeq \forall x\, \neg A$$

Move quantifiers to the front using (provided x is not free in B)

$$(\forall x\, A) \wedge B \simeq \forall x\, (A \wedge B)$$
$$(\forall x\, A) \vee B \simeq \forall x\, (A \vee B)$$

and the similar rules for $\exists$

Skolemization, or Getting Rid of $\exists$

Start with a formula of the form (Can have $k = 0$).

$$\forall x_1\, \forall x_2\, \cdots \forall x_k\, \exists y\, A$$

Choose a fresh k-place function symbol, say f

Delete $\exists y$ and *replace* y by $f(x_1, x_2, \ldots, x_k)$. We get

$$\forall x_1\, \forall x_2\, \cdots \forall x_k\, A[f(x_1, x_2, \ldots, x_k)/y]$$

Repeat until no $\exists$ quantifiers remain

Example of Conversion to Clauses

For proving $\exists x\,[P(x) \rightarrow \forall y\, P(y)]$

$\neg\,[\exists x\,[P(x) \rightarrow \forall y\, P(y)]]$	negated goal
$\forall x\,[P(x) \wedge \exists y\, \neg P(y)]$	conversion to NNF
$\forall x\, \exists y\,[P(x) \wedge \neg P(y)]$	pulling $\exists$ out
$\forall x\,[P(x) \wedge \neg P(f(x))]$	Skolem term $f(x)$
$\{P(x)\}$ $\{\neg P(f(x))\}$	Final clauses

Correctness of Skolemization

The formula $\forall x\, \exists y\, A$ is consistent

$\iff$ it holds in some interpretation $\mathcal{I} = (D, I)$

$\iff$ for all $x \in D$ there is some $y \in D$ such that A holds

$\iff$ some function $\hat{f}$ in $D \rightarrow D$ yields suitable values of y

$\iff$ $A[f(x)/y]$ holds in some $\mathcal{I}'$ extending $\mathcal{I}$ so that f denotes $\hat{f}$

$\iff$ the formula $\forall x\, A[f(x)/y]$ is consistent.

Don't panic if you can't follow this reasoning!

Simplifying the Search for Models

S is satisfiable if even *one* model makes all of its clauses true.

There are *infinitely many* models to consider!

Also many *duplicates*: "states of the USA" and "the integers 1 to 50"

Fortunately, nice models exist.

- They have a *uniform structure* based on the language's *syntax*.
- They satisfy the clauses if any model does.

The Herbrand Universe for a Set of Clauses S

$H_0 \stackrel{\text{def}}{=}$ the set of constants in S *(must be non-empty)*

$H_{i+1} \stackrel{\text{def}}{=} H_i \cup \{f(t_1, \ldots, t_n) \mid t_1, \ldots, t_n \in H_i$ and f is an n-place function symbol in $S\}$

$H \stackrel{\text{def}}{=} \bigcup_{i \geq 0} H_i$ *Herbrand Universe*

H_i contains just the terms with at most i nested function applications.

H consists of the terms in S that contain no variables (*ground* terms).

The Herbrand Semantics of Terms

In an Herbrand model, every constant stands for itself.

Every function symbol stands for a term-forming operation:

f *denotes the function that puts 'f' in front of the given arguments.*

In an Herbrand model, $X + 0$ can never equal X.

Every ground term denotes itself.
This is the promised uniform structure!

The Herbrand Semantics of Predicates

An Herbrand interpretation defines an n-place predicate P to denote a truth-valued function in $H^n \rightarrow \{\mathbf{t}, \mathbf{f}\}$, making $P(t_1, \ldots, t_n)$ true …

- if and only if the *formula* $P(t_1, \ldots, t_n)$ holds in our desired "real" interpretation $\mathcal{I}$ of the clauses.
- Thus, an Herbrand interpretation can imitate *any* other interpretation.

Example of an Herbrand Model

$$\left.\begin{array}{r}\neg even(1)\\ even(2)\\ even(X\cdot Y)\leftarrow even(X), even(Y)\end{array}\right\}\text{clauses}$$

$$H = \{1, 2, 1\cdot 1, 1\cdot 2, 2\cdot 1, 2\cdot 2, 1\cdot(1\cdot 1), \ldots\}$$
$$HB = \{even(1), even(2), even(1\cdot 1), even(1\cdot 2), \ldots\}$$
$$I[even] = \{even(2), even(1\cdot 2), even(2\cdot 1), even(2\cdot 2), \ldots\}$$

(for model where $\cdot$ means product; could instead use sum!)

A Key Fact about Herbrand Interpretations

Let S be a set of clauses.

S is unsatisfiable $\iff$ no Herbrand interpretation satisfies S

- Holds because some Herbrand model mimics every 'real' model
- We must consider only a small class of models
- Herbrand models are syntactic, easily processed by computer

Herbrand's Theorem

Let S be a set of clauses.

S is unsatisfiable $\iff$ there is a finite *unsatisfiable set S' of* ground instances *of clauses of S.*

- **Finite**: we can compute it
- **Instance**: result of substituting for variables
- **Ground**: no variables remain—it's propositional!

Example: S could be $\{P(x)\}$ $\{\neg P(f(y))\}$,
and S' could be $\{P(f(a))\}$ $\{\neg P(f(a))\}$.

Unification

Finding a *common instance* of two terms. Lots of applications:

- **Prolog** and other logic programming languages
- **Theorem proving**: resolution and other procedures
- Tools for reasoning with **equations** or satisfying **constraints**
- Polymorphic type-checking (**ML** and other functional languages)

It is an intuitive generalization of pattern-matching.

Substitutions: A Mathematical Treatment

A substitution is a finite set of *replacements*

$$\theta = [t_1/x_1, \ldots, t_k/x_k]$$

where x_1, ..., x_k are distinct variables and $t_i \neq x_i$.

$$f(t,u)\theta = f(t\theta, u\theta) \qquad \text{(substitution in } \textit{terms}\text{)}$$
$$P(t,u)\theta = P(t\theta, u\theta) \qquad \text{(in } \textit{literals}\text{)}$$
$$\{L_1, \ldots, L_m\}\theta = \{L_1\theta, \ldots, L_m\theta\} \qquad \text{(in } \textit{clauses}\text{)}$$

Composing Substitutions

Composition of ϕ and θ, written $\phi\circ\theta$, satisfies for all terms t

$$t(\phi\circ\theta) = (t\phi)\theta$$

It is defined by (for all relevant x)

$$\phi\circ\theta \stackrel{\text{def}}{=} [\,(x\phi)\theta\,/\,x, \ldots]$$

Consequences include $\theta\circ[] = \theta$, and *associativity*:

$$(\phi\circ\theta)\circ\sigma = \phi\circ(\theta\circ\sigma)$$

Most General Unifiers

θ is a *unifier* of terms t and u if $t\theta = u\theta$.

θ is *more general* than ϕ if $\phi = \theta \circ \sigma$ for some substitution σ.

θ is *most general* if it is more general than every other unifier.

If θ unifies t and u then so does $\theta \circ \sigma$:

$$t(\theta \circ \sigma) = t\theta\sigma = u\theta\sigma = u(\theta \circ \sigma)$$

A most general unifier of $f(a, x)$ and $f(y, g(z))$ is $[a/y, g(z)/x]$.

The common instance is $f(a, g(z))$.

The Unification Algorithm

Represent terms by *binary trees*.

Each term is a *Variable* $x, y \ldots$, *Constant* $a, b \ldots$, or *Pair* (t, t')

SKETCH OF THE ALGORITHM.

Constants do not unify with different Constants or with Pairs.

Variable x and term t: if x occurs in t, FAIL. Otherwise, unifier is $[t/x]$.

Cannot unify $f(\cdots x \cdots)$ with x!

The Unification Algorithm: The Case of Two Pairs

$\theta \circ \theta'$ unifies (t, t') with (u, u')

if θ unifies t with u and θ' unifies $t'\theta$ with $u'\theta$.

We unify the left sides, then the right sides.

In an implementation, substitutions are formed by *updating pointers*.

Composition happens automatically as more pointers are updated.

Mathematical Justification

It's easy to check that $\theta \circ \theta'$ unifies (t, t') with (u, u'):

$$\begin{aligned}
(t, t')(\theta \circ \theta') &= (t, t')\theta\theta' && \text{definition of substitution} \\
&= (t\theta\theta', t'\theta\theta') && \text{substituting into the pair} \\
&= (u\theta\theta', t'\theta\theta') && t\theta = u\theta \\
&= (u\theta\theta', u'\theta\theta') && t'\theta\theta' = u'\theta\theta' \\
&= (u, u')(\theta \circ \theta') && \text{definition of substitution}
\end{aligned}$$

In fact $\theta \circ \theta'$ is even a most general unifier, if θ and θ' are!

Four Unification Examples

$f(x, b)$	$f(x, x)$	$f(x, x)$	$j(x, x, z)$
$f(a, y)$	$f(a, b)$	$f(y, g(y))$	$j(w, a, h(w))$
$f(a, b)$	**None**	**None**	$j(a, a, h(a))$
$[a/x, b/y]$	**Fail**	**Fail**	$[a/w, a/x, h(a)/z]$

Remember, the output is a *substitution*.

The algorithm naturally yields a *most general* unifier.

Theorem-Proving Example 1

$$(\exists y\, \forall x\, R(x, y)) \to (\forall x\, \exists y\, R(x, y))$$

After negation, the clauses are $\{R(x, a)\}$ and $\{\neg R(b, y)\}$.

The literals $R(x, a)$ and $R(b, y)$ have unifier $[b/x, a/y]$.

We have the contradiction $R(b, a)$ and $\neg R(b, a)$.

THE THEOREM IS PROVED BY CONTRADICTION!

Theorem-Proving Example 2

$$(\forall x \, \exists y \, R(x, y)) \to (\exists y \, \forall x \, R(x, y))$$

After negation, the clauses are $\{R(x, f(x))\}$ and $\{\neg R(g(y), y)\}$.

The literals $R(x, f(x))$ and $R(g(y), y)$ are not unifiable.

(They fail the *occurs check*.)

We can't get a contradiction. FORMULA IS NOT A THEOREM!

Variations on Unification

Efficient unification algorithms: near-linear time

Indexing & Discrimination networks: fast retrieval of a unifiable term

Associative/commutative unification

- *Example*: unify $a + (y + c)$ with $(c + x) + b$, get $[a/x, b/y]$
- Algorithm is very complicated
- The number of unifiers can be exponential

Unification in many other theories (often undecidable!)

The Binary Resolution Rule

$$\frac{\{B, A_1, \ldots, A_m\} \qquad \{\neg D, C_1, \ldots, C_n\}}{\{A_1, \ldots, A_m, C_1, \ldots, C_n\}\sigma} \qquad \textit{provided } B\sigma = D\sigma$$

(σ is a *most general* unifier of B and D.)

First, *rename variables apart* in the clauses! For example, given

$$\{P(x)\} \quad \text{and} \quad \{\neg P(g(x))\},$$

rename x in one of the clauses. (Otherwise, unification would fail.)

The Factoring Rule

This inference collapses unifiable literals *in one clause*:

$$\frac{\{B_1, \ldots, B_k, A_1, \ldots, A_m\}}{\{B_1, A_1, \ldots, A_m\}\sigma} \qquad \textit{provided } B_1\sigma = \cdots = B_k\sigma$$

Example: Prove $\forall x \, \exists y \, \neg(P(y, x) \leftrightarrow \neg P(y, y))$

The clauses are	$\{\neg P(y, a), \neg P(y, y)\}$	$\{P(y, y), P(y, a)\}$
Factoring yields	$\{\neg P(a, a)\}$	$\{P(a, a)\}$

Resolution yields the empty clause!

A Non-Trivial Proof

$$\exists x \, [P \to Q(x)] \land \exists x \, [Q(x) \to P] \to \exists x \, [P \leftrightarrow Q(x)]$$

Clauses are

$\{P, \neg Q(b)\}$ $\{P, Q(x)\}$ $\{\neg P, \neg Q(x)\}$ $\{\neg P, Q(a)\}$

Resolve $\{P, \underline{\neg Q(b)}\}$ with $\{P, \underline{Q(x)}\}$	getting $\{P, P\}$
Factor $\{P, P\}$	getting $\{P\}$
Resolve $\{\neg P, \underline{\neg Q(x)}\}$ with $\{\neg P, \underline{Q(a)}\}$	getting $\{\neg P, \neg P\}$
Factor $\{\neg P, \neg P\}$	getting $\{\neg P\}$
Resolve $\{P\}$ with $\{\neg P\}$	getting $\square$

What About Equality?

In theory, it's enough to add the *equality axioms*:

- The *reflexive*, *symmetric* and *transitive* laws.
- *Substitution* laws like $\{x \neq y, f(x) = f(y)\}$ for each f.
- *Substitution* laws like $\{x \neq y, \neg P(x), P(y)\}$ for each P.

In practice, we need something special: the *paramodulation rule*

$$\frac{\{B[t'], A_1, \ldots, A_m\} \qquad \{t = u, C_1, \ldots, C_n\}}{\{B[u], A_1, \ldots, A_m, C_1, \ldots, C_n\}\sigma} \qquad \textit{(if } t\sigma = t'\sigma\textit{)}$$

Prolog Clauses

Prolog clauses have a restricted form, with *at most one* positive literal.

The *definite clauses* form the program. Procedure B with body "commands" $A_1, \ldots, A_m$ is

$$B \leftarrow A_1, \ldots, A_m$$

The single *goal clause* is like the "execution stack", with say m tasks left to be done.

$$\leftarrow A_1, \ldots, A_m$$

Prolog Execution

Linear resolution:

- Always resolve some program clause with the goal clause.
- The result becomes the new goal clause.

Try the program clauses in *left-to-right* order.

Solve the goal clause's literals in *left-to-right* order.

Use *depth-first search*. (Performs *backtracking*, using little space.)

Do unification without *occurs check*. (UNSOUND, but needed for speed)

A (Pure) Prolog Program

```
parent(elizabeth,charles).
parent(elizabeth,andrew).

parent(charles,william).
parent(charles,henry).

parent(andrew,beatrice).
parent(andrew,eugenia).

grand(X,Z) :- parent(X,Y), parent(Y,Z).
cousin(X,Y) :- grand(Z,X), grand(Z,Y).
```


Prolog Execution

```
                                         :- cousin(X,Y).
                           :- grand(Z1,X), grand(Z1,Y).
         :- parent(Z1,Y2), parent(Y2,X), grand(Z1,Y).
*    :- parent(charles,X), grand(elizabeth,Y).
X=william                    :- grand(elizabeth,Y).
          :- parent(elizabeth,Y5), parent(Y5,Y).
*                             :- parent(andrew,Y).
Y=beatrice                                    :- □.
```

`*` = backtracking choice point

16 solutions including `cousin(william,william)`

and `cousin(william,henry)`

Another FOL Proof Procedure: Model Elimination

A Prolog-like method to run on fast Prolog architectures.

Contrapositives: treat clause $\{A_1, \ldots, A_m\}$ like the m clauses

$$A_1 \leftarrow \neg A_2, \ldots, \neg A_m$$
$$A_2 \leftarrow \neg A_3, \ldots, \neg A_m, \neg A_1$$
$$\vdots$$
$$A_m \leftarrow \neg A_1, \ldots, \neg A_{m-1}$$

Extension rule: when proving goal P, assume $\neg P$.

A Survey of Automatic Theorem Provers

Saturation (that is, resolution): E, Gandalf, SPASS, Vampire, ...

Higher-Order Logic: TPS, LEO, LEO-II

Model Elimination: Prolog Technology Theorem Prover, SETHEO

Parallel ME: PARTHENON, PARTHEO

Tableau (sequent) based: LeanTAP, 3TAP, ...

BDDs: Binary Decision Diagrams

A *canonical form* for boolean expressions: decision trees with sharing.

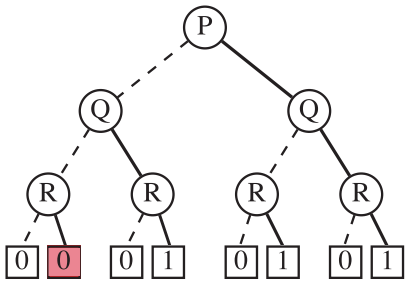
- *ordered* propositional symbols (the *variables*)
- *sharing* of identical subtrees
- *hashing* and other optimisations

Detects if a formula is tautologous (=1) or inconsistent (=0).

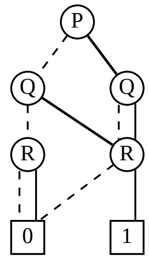
Exhibits *models* (paths to 1) if the formula is satisfiable.

Excellent for verifying digital circuits, with many other applications.

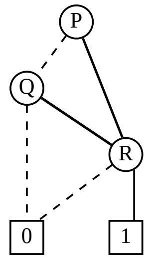
Decision Diagram for $(P \vee Q) \wedge R$



Converting a Decision Diagram to a BDD



No duplicates



No redundant tests

Building BDDs Efficiently

Do not construct the full binary tree!

Do not expand $\rightarrow$, $\leftrightarrow$, $\oplus$ (exclusive OR) to other connectives!!

- Recursively convert operands to BDDs.
- Combine operand BDDs, respecting the ordering and sharing.
- Delete redundant variable tests.

Canonical Form Algorithm

To convert $Z \wedge Z'$, where Z and Z' are already BDDs:

Trivial if either operand is 1 or 0.

Let $Z = \textbf{if}(P, X, Y)$ and $Z' = \textbf{if}(P', X', Y')$

- If $P = P'$ then recursively convert $\textbf{if}(P,\ X \wedge X',\ Y \wedge Y')$.
- If $P < P'$ then recursively convert $\textbf{if}(P,\ X \wedge Z',\ Y \wedge Z')$.
- If $P > P'$ then recursively convert $\textbf{if}(P',\ Z \wedge X',\ Z \wedge Y')$.

Canonical Forms of Other Connectives

$Z \vee Z'$, $Z \rightarrow Z'$ and $Z \leftrightarrow Z'$ are converted to BDDs similarly.

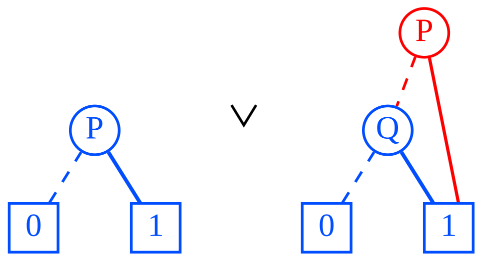
Some cases, like $Z \rightarrow 0$ and $Z \leftrightarrow 0$, reduce to negation.

Here is how to convert $\neg Z$, where Z is a BDD:

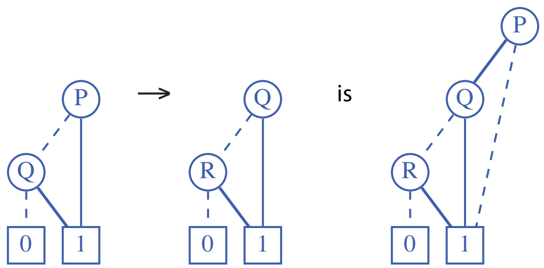
- If $Z = \textbf{if}(P, X, Y)$ then recursively convert $\textbf{if}(P,\ \neg X,\ \neg Y)$.
- if $Z = 1$ then return 0, and if $Z = 0$ then return 1.

(In effect we copy the BDD but exchange the 1 and 0 at the bottom.)

Canonical Form (that is, BDD) of $P \vee Q$



Canonical Form of $P \vee Q \to Q \vee R$



Optimisations

Never build the same BDD twice, but share pointers.

Advantages:

- If $X \simeq Y$, then the addresses of X and Y are equal.
- Can see if **if**(P, X, Y) is redundant by checking if $X = Y$.
- Can quickly simplify special cases like $X \wedge X$.

Never convert $X \wedge Y$ *twice*, but keep a hash table of known canonical forms. This prevents redundant computations.

Final Observations

The variable ordering is crucial. Consider this formula:

$$(P_1 \wedge Q_1) \vee \cdots \vee (P_n \wedge Q_n)$$

A *good ordering* is $P_1 < Q_1 < \cdots < P_n < Q_n$: the BDD is linear.

With $P_1 < \cdots < P_n < Q_1 < \cdots < Q_n$, the BDD is EXPONENTIAL.

Many digital circuits have small BDDs: adders, but not multipliers.

BDDs can solve problems in hundreds of variables.

The general case remains hard (it is NP-complete).

Modal Operators

W: set of *possible worlds* (machine states, future times, . . .)

R: *accessibility relation* between worlds

(W, R) is called a *modal frame*

$\Box A$ means A is *necessarily true* }
$\Diamond A$ means A is *possibly true* } in all worlds **accessible** from **here**

$\neg\Diamond A \simeq \Box\neg A$ $\qquad$ A *cannot be true* $\iff$ A *must be false*

Semantics of Propositional Modal Logic

For a particular frame (W, R)

An *interpretation* I maps the propositional letters to *subsets* of W

$w \Vdash A$ means A *is true in world* w

$w \Vdash P \iff w \in I(P)$

$w \Vdash A \wedge B \iff w \Vdash A$ and $w \Vdash B$

$w \Vdash \Box A \iff v \Vdash A$ for all v such that $R(w, v)$

$w \Vdash \Diamond A \iff v \Vdash A$ for some v such that $R(w, v)$

Truth and Validity in Modal Logic

For a particular frame (W, R), and interpretation I

$w \Vdash A$	means A is true in world w
$\models_{W,R,I} A$	means $w \Vdash A$ for all w in W
$\models_{W,R} A$	means $w \Vdash A$ for all w and all I

$\models A$ means $\models_{W,R} A$ for all frames; A is *universally valid*

. . . but typically we constrain R to be, say, **transitive**.

All propositional tautologies are universally valid!

A Hilbert-Style Proof System for K

Extend your favourite propositional proof system with

Dist $\quad \Box(A \to B) \to (\Box A \to \Box B)$

Inference Rule: *Necessitation*

$$\frac{A}{\Box A}$$

Treat $\Diamond$ as a *definition*

$$\Diamond A \stackrel{\text{def}}{=} \neg\Box\neg A$$

Variant Modal Logics

Start with pure modal logic, which is called K

Add *axioms* to constrain the accessibility relation:

T	$\Box A \to A$	*(reflexive)*	logic T
4	$\Box A \to \Box\Box A$	*(transitive)*	logic $S4$
B	$A \to \Box\Diamond A$	*(symmetric)*	logic $S5$

And countless others!

We mainly look at $S4$, which resembles a logic of time.

Extra Sequent Calculus Rules for $S4$

$$\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\ (\Box l) \qquad \frac{\Gamma^* \Rightarrow \Delta^*, A}{\Gamma \Rightarrow \Delta, \Box A}\ (\Box r)$$

$$\frac{A, \Gamma^* \Rightarrow \Delta^*}{\Diamond A, \Gamma \Rightarrow \Delta}\ (\Diamond l) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \Diamond A}\ (\Diamond r)$$

$\Gamma^* \stackrel{\text{def}}{=} \{\Box B \mid \Box B \in \Gamma\}$ Erase *non-*$\Box$ assumptions.

$\Delta^* \stackrel{\text{def}}{=} \{\Diamond B \mid \Diamond B \in \Delta\}$ Erase *non-*$\Diamond$ goals!

A Proof of the Distribution Axiom

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \Rightarrow B, A} \qquad \overline{B, A \Rightarrow B}}{A \to B, A \Rightarrow B}\ (\to l)}{A \to B, \Box A \Rightarrow B}\ (\Box l)}{\Box(A \to B), \Box A \Rightarrow B}\ (\Box l)}{\Box(A \to B), \Box A \Rightarrow \Box B}\ (\Box r)}{}$$

And thus $\Box(A \to B) \to (\Box A \to \Box B)$

Must apply $(\Box r)$ first!

Part of an "Operator String Equivalence"

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\Diamond A \Rightarrow \Diamond A}}{\Box\Diamond A \Rightarrow \Diamond A}\ (\Box l)}{\Diamond\Box\Diamond A \Rightarrow \Diamond A}\ (\Diamond l)}{\Box\Diamond\Box\Diamond A \Rightarrow \Diamond A}\ (\Box l)}{\Box\Diamond\Box\Diamond A \Rightarrow \Box\Diamond A}\ (\Box r)$$

In fact, $\Box\Diamond\Box\Diamond A \simeq \Box\Diamond A$ also $\Box\Box A \simeq \Box A$

The $S4$ operator strings are

$\Box \quad \Diamond \quad \Box\Diamond \quad \Diamond\Box \quad \Box\Diamond\Box \quad \Diamond\Box\Diamond$

Two Failed Proofs

$$\dfrac{\dfrac{\Rightarrow A}{\Rightarrow \Diamond A}\,(\Diamond r)}{A \Rightarrow \Box\Diamond A}\,(\Box r)$$

$$\dfrac{\dfrac{B \Rightarrow A \land B}{B \Rightarrow \Diamond(A \land B)}\,(\Diamond r)}{\Diamond A, \Diamond B \Rightarrow \Diamond(A \land B)}\,(\Diamond l)$$

Can extract a countermodel from the proof attempt

Simplifying the Sequent Calculus

7 connectives *(or 9 for modal logic)*:

$$\neg \quad \land \quad \lor \quad \to \quad \leftrightarrow \quad \forall \quad \exists \qquad (\Box \quad \Diamond)$$

Left and right: so 14 rules *(or 18)* plus basic sequent, cut

Idea! Work in **Negation Normal Form**

Fewer connectives: $\land \quad \lor \quad \forall \quad \exists \qquad (\Box \quad \Diamond)$

Sequents need *one side only!*

Tableau Calculus: Left-Only

$$\dfrac{}{\neg A, A, \Gamma \Rightarrow}\ \text{(basic)} \qquad \dfrac{\neg A, \Gamma \Rightarrow \qquad A, \Gamma \Rightarrow}{\Gamma \Rightarrow}\ \text{(cut)}$$

$$\dfrac{A, B, \Gamma \Rightarrow}{A \land B, \Gamma \Rightarrow}\ (\land l) \qquad \dfrac{A, \Gamma \Rightarrow \qquad B, \Gamma \Rightarrow}{A \lor B, \Gamma \Rightarrow}\ (\lor l)$$

$$\dfrac{A[t/x], \Gamma \Rightarrow}{\forall x\, A, \Gamma \Rightarrow}\ (\forall l) \qquad \dfrac{A, \Gamma \Rightarrow}{\exists x\, A, \Gamma \Rightarrow}\ (\exists l)$$

Rule $(\exists l)$ holds *provided* x is not free in the conclusion!

Tableau Rules for $S4$

$$\dfrac{A, \Gamma \Rightarrow}{\Box A, \Gamma \Rightarrow}\ (\Box l) \qquad \dfrac{A, \Gamma^* \Rightarrow}{\Diamond A, \Gamma \Rightarrow}\ (\Diamond l)$$

$\Gamma^* \stackrel{\text{def}}{=} \{\Box B \mid \Box B \in \Gamma\}$ Erase non-$\Box$ assumptions

From 14 *(or 18)* rules to 4 *(or 6)*

Left-side only system uses **proof by contradiction**

Right-side only system is an exact *dual*

Tableau Proof of $\forall x\,(P \to Q(x)) \Rightarrow P \to \forall y\, Q(y)$

Move the right-side formula to the left and convert to NNF:

$P \land \exists y\, \neg Q(y),\ \forall x\,(\neg P \lor Q(x)) \Rightarrow$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P, \neg Q(y), \neg P \Rightarrow} \qquad \dfrac{}{P, \neg Q(y), Q(y) \Rightarrow}}{P, \neg Q(y), \neg P \lor Q(y) \Rightarrow}\,(\lor l)}{P, \neg Q(y), \forall x\,(\neg P \lor Q(x)) \Rightarrow}\,(\forall l)}{P, \exists y\, \neg Q(y), \forall x\,(\neg P \lor Q(x)) \Rightarrow}\,(\exists l)}{P \land \exists y\, \neg Q(y), \forall x\,(\neg P \lor Q(x)) \Rightarrow}\,(\land l)$$

The Free-Variable Tableau Calculus

Rule $(\forall l)$ now inserts a **new** free variable:

$$\dfrac{A[z/x], \Gamma \Rightarrow}{\forall x\, A, \Gamma \Rightarrow}\ (\forall l)$$

Let unification instantiate *any free variable*

In $\neg A, B, \Gamma \Rightarrow$ try unifying A with B to make a basic sequent

Updating a variable affects *entire proof tree*

What about rule $(\exists l)$? DO NOT USE IT! Instead, *Skolemize!*

Skolemization from NNF

Don't pull quantifiers out! Skolemize

$[\forall y\, \exists z\, Q(y,z)] \land \exists x\, P(x)$ to $[\forall y\, Q(y, f(y))] \land P(a)$

It's better to push quantifiers in (called **miniscoping**)

Example: proving $\exists x\, \forall y\, [P(x) \to P(y)]$:

Negate; convert to NNF:	$\forall x\, \exists y\, [P(x) \land \neg P(y)]$
Push in the $\exists y$:	$\forall x\, [P(x) \land \exists y\, \neg P(y)]$
Push in the $\forall x$:	$(\forall x\, P(x)) \land (\exists y\, \neg P(y))$
Skolemize:	$\forall x\, P(x) \land \neg P(a)$

Free-Variable Tableau Proof of $\exists x\, \forall y\, [P(x) \to P(y)]$

$$\begin{array}{r l}
y \mapsto f(z) & \\ \hline
P(y),\ \neg P(f(y)),\ P(z),\ \neg P(f(z)) \Rightarrow & \text{(basic)} \\ \hline
P(y),\ \neg P(f(y)),\ P(z) \land \neg P(f(z)) \Rightarrow & (\land l) \\ \hline
P(y),\ \neg P(f(y)),\ \forall x\, [P(x) \land \neg P(f(x))] \Rightarrow & (\forall l) \\ \hline
P(y) \land \neg P(f(y)),\ \forall x\, [P(x) \land \neg P(f(x))] \Rightarrow & (\land l) \\ \hline
\forall x\, [P(x) \land \neg P(f(x))] \Rightarrow & (\forall l)
\end{array}$$

Unification chooses the term for $(\forall l)$

A Failed Proof

Try to prove $\forall x\, [P(x) \lor Q(x)] \Rightarrow \forall x\, P(x) \lor \forall x\, Q(x)$

NNF: $\exists x\, \neg P(x) \land \exists x\, \neg Q(x),\ \forall x\, [P(x) \lor Q(x)] \Rightarrow$

Skolemize: $\neg P(a) \land \neg Q(b),\ \forall x\, [P(x) \lor Q(x)] \Rightarrow$

$$\dfrac{\dfrac{\dfrac{\dfrac{y \mapsto a}{\neg P(a),\ \neg Q(b),\ P(y) \Rightarrow} \qquad \dfrac{y \mapsto b???}{\neg P(a),\ \neg Q(b),\ Q(y) \Rightarrow}}{\neg P(a),\ \neg Q(b),\ P(y) \lor Q(y) \Rightarrow}\,(\lor l)}{\neg P(a),\ \neg Q(b),\ \forall x\, [P(x) \lor Q(x)] \Rightarrow}\,(\forall l)}{\neg P(a) \land \neg Q(b),\ \forall x\, [P(x) \lor Q(x)] \Rightarrow}\,(\land l)$$

The World's Smallest Theorem Prover?

```
prove((A,B),UnExp,Lits,FreeV,VarLim) :- !,          and
        prove(A,[B|UnExp],Lits,FreeV,VarLim).
prove((A;B),UnExp,Lits,FreeV,VarLim) :- !,           or
        prove(A,UnExp,Lits,FreeV,VarLim),
        prove(B,UnExp,Lits,FreeV,VarLim).
prove(all(X,Fml),UnExp,Lits,FreeV,VarLim) :- !,      forall
        \+ length(FreeV,VarLim),
        copy_term((X,Fml,FreeV),(X1,Fml1,FreeV)),
        append(UnExp,[all(X,Fml)],UnExp1),
        prove(Fml1,UnExp1,Lits,[X1|FreeV],VarLim).
prove(Lit,_,[L|Lits],_,_) :-                         literals; negation
        (Lit = -Neg; -Lit = Neg) ->
        (unify(Neg,L); prove(Lit,[],Lits,_,_)).
prove(Lit,[Next|UnExp],Lits,FreeV,VarLim) :-
next formula
        prove(Next,UnExp,[Lit|Lits],FreeV,VarLim).
```