Categorical theory and logic
Exercise Sheet 3

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Suggestion: In questions 4, 8, 9, 10, 11, 12 it may be helpful to assume $\mathcal{C} = \mathbf{Set}$, to get your bearings.

1. Show that the category **Poset** of posets and monotone functions is a cartesian closed category.
2. Let X be a set. The discrete poset on X has the following partial order: $x \leq y$ iff $x = y$.
   (a) Show that the construction of discrete posets extends to a functor $D : \mathbf{Set} \to \mathbf{Poset}$.
   (b) Show that this functor has a right adjoint, $U : \mathbf{Poset} \to \mathbf{Set}$.
3. Recall that $\hat{\Sigma}$ is the category whose objects are functions and whose morphisms are commuting squares. Consider the following functor $F_0 : \mathbf{Set} \to \hat{\Sigma}$: it takes a set X to the empty function $(\emptyset \to X)$, and a function $f : X \to Y$ to the square

$$\begin{tikzcd} \emptyset \arrow[r, "\mathrm{id}"] \arrow[d, "!"'] & \emptyset \arrow[d, "!"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

   (a) Show that this functor has a right adjoint, a functor $F_1 : \hat{\Sigma} \to \mathbf{Set}$.
   (b) Show that $F_1 : \hat{\Sigma} \to \mathbf{Set}$ has a right adjoint, $F_2 : \mathbf{Set} \to \hat{\Sigma}$.
   (c) Does F_3 have a right adjoint, F_4? If so does F_4 have a right adjoint, F_5? If so does F_5 have a right adjoint? etc
4. Consider a cartesian closed category, $\mathcal{C}$. Let S be an object of $\mathcal{C}$. Define the category $\mathcal{C}_S$ as follows: the objects are the objects of $\mathcal{C}$; a morphism $(X \to Y)$ in $\mathcal{C}_S$ is given by a morphism $((X \times S) \to (Y \times S))$ in $\mathcal{C}$. [Here is some intuition: the morphisms of $\mathcal{C}_S$ are "stateful computations", that take an argument and an initial state, and return a result and a final state.]
   (a) Finish describing the category $\mathcal{C}_S$.
   (b) Describe a functor $\mathcal{C} \to \mathcal{C}_S$ which is identity on objects.
   (c) Show that the functor has a right adjoint.
5. Consider a functor $F : \mathcal{C} \to \mathcal{D}$ that has a right adjoint, $G : \mathcal{D} \to \mathcal{C}$.

   If G is full and faithful, then it is called a 'reflection'. Which of the functors F_0, F_1... in Question 3 are reflections?

   Show that G is full and faithful if and only if the counit $\{\varepsilon_X : F(GX) \to X\}_{X \in \mathrm{ob}(\mathcal{D})}$ is an isomorphism. [It's good to know about this last part, but don't worry about proving it if you find it tricky. It's better to move on to the next page.]

6. What is a monomorphism in the category of posets and monotone functions?
7. Show that the category of posets and monotone functions has pullbacks.
8. Suppose that a category $\mathcal{C}$ has a terminal object. Describe an isomorphism of categories $\mathcal{C} \to \mathcal{C}/1$.
9. Consider morphisms $f : X \to Y$, $g : Y \to Z$ in a category. Which of the following statements are true? Either prove it, or find a counter-example in the category of sets.
   (a) If f and g are monomorphisms, so is gf.
   (b) If f and gf are monomorphisms, so is g.
   (c) If g and gf are monomorphisms, so is f.
10. Consider a morphism $f : X \to Y$ in a category. Show that the following square is a pullback if and only if f is a monomorphism.

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

11. Consider the following commuting squares in a category.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[r, "g"] \arrow[d, "j"] & C \arrow[d, "k"] \\ D \arrow[r, "m"'] & E \arrow[r, "n"'] & F \end{tikzcd}$$

   Which of the following statements are true? Either prove it, or find a counter-example in the category of sets.
   (a) If both inner squares are pullbacks, so is the outer square.
   (b) If the right-hand square and the outer rectangle are pullbacks, so is the left-hand square.
   (c) If the left-hand square and the outer rectangle are pullbacks, so is the right-hand square.
12. Consider a category $\mathcal{C}$ with products. For every object A there is always a morphism, $\delta_A = (\mathrm{id}_A, \mathrm{id}_A) : A \to A \times A$. [It is "the meaning of equality".]
   (a) Show that δ_A is always a monomorphism.
   (b) Post-composing with δ_A yields a functor $\Sigma_{\delta_A} : \mathcal{C}/A \to \mathcal{C}/(A \times A)$.
   Show that $\mathcal{C}/A$ has finite products if and only if Σ_{δ_A} has a right adjoint.