2009 Paper 9 Question 6

(*a*) (*i*) Define the notion of *contextual equivalence* in PCF. [2 marks]

(*ii*) Consider the following two closed PCF terms of type $nat \to nat \to nat$.

$$
\begin{aligned}
F \stackrel{\text{def}}{=} \mathbf{fix}\big(\ &\mathbf{fn}\, f : nat \to nat \to nat.\ \mathbf{fn}\, x : nat.\ \mathbf{fn}\ y : nat.\\
&\mathbf{if\ zero}(x)\ \mathbf{then\ 0}\\
&\mathbf{else\ if\ zero}(y)\ \mathbf{then\ 0}\\
&\mathbf{else\ succ}\big(\ f\,(\mathbf{pred}\ x)\,(\mathbf{pred}\ y)\big)\ \big)
\end{aligned}
$$

$$
\begin{aligned}
G \stackrel{\text{def}}{=} \mathbf{fix}\big(\ &\mathbf{fn}\, g : nat \to nat \to nat.\ \mathbf{fn}\, x : nat.\ \mathbf{fn}\ y : nat.\\
&\mathbf{if\ zero}(y)\ \mathbf{then\ 0}\\
&\mathbf{else\ if\ zero}(x)\ \mathbf{then\ 0}\\
&\mathbf{else\ succ}\big(\ g\,(\mathbf{pred}\ x)\,(\mathbf{pred}\ y)\big)\ \big)
\end{aligned}
$$

Are F and G contextually equivalent? Justify your answer. [5 marks]

Contextual equivalence of PCF terms

Given PCF terms M_1, M_2, PCF type τ, and a type environment Γ, the relation $\boxed{\Gamma \vdash M_1 \cong_{\mathrm{ctx}} M_2 : \tau}$ is defined to hold iff

- Both the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold.
- For all PCF contexts $\mathcal{C}$ for which $\mathcal{C}[M_1]$ and $\mathcal{C}[M_2]$ are closed terms of type γ, *where* $\gamma = nat$ *or* $\gamma = bool$, and for all values $V : \gamma$,

$$\mathcal{C}[M_1] \Downarrow_\gamma V \;\Leftrightarrow\; \mathcal{C}[M_2] \Downarrow_\gamma V.$$

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\begin{aligned}
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&\qquad\qquad \mathbf{if\ zero}(y)\ \mathbf{then\ 0}\\
&\qquad\qquad \mathbf{else\ if\ zero}(x)\ \mathbf{then\ 0}\\
&\qquad\qquad \mathbf{else\ succ}\big(\ g\,(\mathbf{pred}\ x)\,(\mathbf{pred}\ y)\,\big)\ \big)
\end{aligned}
$$

Are F and G contextually equivalent? Justify your answer. [5 marks]

IDEA : $\begin{cases} F\,0\,\Omega & \text{evaluates to } 0 \\ G\,0\,\Omega & \text{does not evaluate to any value} \end{cases}$

(where $\Omega \stackrel{\text{def}}{=} \mathbf{fix}(\mathbf{fn}\, x : nat .\, x)$)

so context $\mathcal{C} \stackrel{\text{def}}{=} (-)\,0\,\Omega$ satisfies

$$\begin{cases} \mathcal{C}[F] \Downarrow_{nat} 0 \\ \mathcal{C}[G] \not\Downarrow_{nat} \end{cases}$$

and hence

$$\boxed{F \,\&\, G \text{ are } \underline{\text{not}} \text{ contextually equivalent}}$$

$$\boxed{\begin{array}{ll} F\,0\,\Omega & \text{evaluates to } 0 \\ G\,0\,\Omega & \text{does not evaluate to any value} \end{array}}$$

Easiest (?) way to see this is via the transition relation for PCF (Fig. 4, page 61):

$F\,0\,\Omega$ evaluates to 0

$G\,0\,\Omega$ does not evaluate to any value

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$$F\,0\,\Omega \to^{*} \begin{array}{l}\text{if zero}(0)\text{ then } 0 \text{ else}\\ \text{if zero}(\Omega)\text{ then } 0 \text{ else}\\ \text{succ}(F(\text{pred}(0))(\text{pred}(\Omega)))\end{array} \to 0$$

$$\boxed{\begin{array}{ll} F\,0\,\Omega & \text{evaluates to } 0 \\ G\,0\,\Omega & \text{does not evaluate to any value} \end{array}}$$

Easiest (?) way to see this is via the transition relation for PCF (Fig. 4, page 61):

$$F\,0\,\Omega \to^* \begin{array}{l} \text{if zero}(0) \text{ then } 0 \text{ else} \\ \text{if zero}(\Omega) \text{ then } 0 \text{ else} \\ \text{succ}(F(\text{pred}(0))(\text{pred}(\Omega))) \end{array} \to 0$$

$$G\,0\,\Omega \to^* \begin{array}{l} \text{if zero}(\Omega) \text{ then } 0 \text{ else} \\ \text{if zero}(0) \text{ then } 0 \text{ else} \\ \text{succ}(G(\text{pred}(0))(\text{pred}(\Omega))) \end{array}$$

$$\boxed{\begin{array}{l} \text{because} \\ \Omega \to \Omega \end{array}}$$

$$\boxed{\begin{array}{ll} \mathrm{F}\,0\,\Omega & \text{evaluates to } 0 \\ \mathrm{G}\,0\,\Omega & \text{does not evaluate to any value} \end{array}}$$

Another way to see this is via the evaluation relation for PCF (Fig. 3, page 59):

$F\,0\,\Omega$ evaluates to 0

$G\,0\,\Omega$ does not evaluate to any value

Yet another way to see this is via the denotational semantics of PCF (p69 et seq.):

$[\![F]\!] = \mathit{fix}(\Phi)$ where $\Phi : (\underset{f}{N_\perp \to N_\perp \to N_\perp}) \to (\underset{x}{N_\perp} \to \underset{y}{N_\perp} \to N_\perp)$

$$\Phi(f)(x)(y) = \begin{cases} \perp & \text{if } x = \perp \text{ or } (x > 0 \;\&\; y = \perp) \\ 0 & \text{if } x = 0 \text{ or } (x > 0 \;\&\; y = 0) \\ f(x-1)(y-1)+1 & \text{if } x > 0 \;\&\; y > 0 \end{cases}$$

$$\boxed{\begin{array}{ll} F\ 0\ \Omega & \text{evaluates to } 0 \\ G\ 0\ \Omega & \text{does not evaluate to any value} \end{array}}$$

Yet another way to see this is via the denotational semantics of PCF (p69 et seq.):

$$[\![F]\!] = fix(\Phi) \text{ where } \Phi : \underbrace{(N_\perp \to N_\perp \to N_\perp)}_{f} \to (\underset{x}{N_\perp} \to \underset{y}{N_\perp} \to N_\perp)$$

$$\boxed{\Phi(f)(x)(y) = \begin{cases} \perp & \text{if } x = \perp \text{ or } (x > 0 \;\&\; y = \perp) \\ 0 & \text{if } x = 0 \text{ or } (x > 0 \;\&\; y = 0) \\ f(x-1)(y-1)+1 & \text{if } x > 0 \;\&\; y > 0 \end{cases}}$$

so $[\![F\ 0\ \Omega]\!] = [\![F]\!](0)(\perp) = \Phi([\![F]\!])(0)(\perp) = 0$

$$\boxed{\begin{array}{ll} F\,0\,\Omega & \text{evaluates to } 0 \\ G\,0\,\Omega & \text{does not evaluate to any value} \end{array}}$$

Yet another way to see this is via the denotational semantics of PCF (p69 et seq.):

$[\![G]\!] = fix(\Psi)$ where $\Psi : (\underset{f}{\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp}) \to (\underset{x}{\mathbb{N}_\perp} \to \underset{y}{\mathbb{N}_\perp} \to \mathbb{N}_\perp)$

$$\boxed{\Psi(f)(x)(y) = \begin{cases} \perp & \text{if } y = \perp \text{ or } (y>0 \,\&\, x=\perp) \\ 0 & \text{if } y=0 \text{ or } (y>0 \,\&\, x=0) \\ f(x-1)(y-1)+1 & \text{if } x>0 \,\&\, y>0 \end{cases}}$$

so $[\![G\,0\,\Omega]\!] = [\![G]\!](0)(\perp) = \Psi([\![G]\!])(0)(\perp) = \perp$

$F\,0\,\Omega$ evaluates to 0

$G\,0\,\Omega$ does not evaluate to any value

Yet another way to see this is via the denotational semantics of PCF (p69 et seq.):

So $[\![F\,0\,\Omega]\!] = 0 = [\![0]\!]$

$[\![G\,0\,\Omega]\!] = \perp \neq [\![0]\!]$

By the $\left\{\begin{array}{l}\text{adequacy property}\\ \text{soundness property}\end{array}\right.$ we have $\left\{\begin{array}{l}F\,0\,\Omega \Downarrow_{nat} 0\\ G\,0\,\Omega \not\Downarrow_{nat} 0\end{array}\right.$

PCF denotational semantics — aims

- PCF types $\tau \;\mapsto\;$ domains $[\![\tau]\!]$.
- Closed PCF terms $M : \tau \;\mapsto\;$ elements $[\![M]\!] \in [\![\tau]\!]$.

  Denotations of open terms will be continuous functions.
- Compositionality.

  In particular: $[\![M]\!] = [\![M']\!] \;\Rightarrow\; [\![\mathcal{C}[M]]\!] = [\![\mathcal{C}[M']]\!]$.
- Soundness.

  For any type τ, $M \Downarrow_\tau V \;\Rightarrow\; [\![M]\!] = [\![V]\!]$.
- Adequacy.

  For $\tau = bool$ or nat, $[\![M]\!] = [\![V]\!] \in [\![\tau]\!] \;\Longrightarrow\; M \Downarrow_\tau V$.

$F\,0\,\Omega$ evaluates to 0

$G\,0\,\Omega$ does not evaluate to any value

Yet another way to see this is via the denotational semantics of PCF (p69 et seq.):

So $[\![F\,0\,\Omega]\!] = 0 = [\![0]\!]$

$[\![G\,0\,\Omega]\!] = \bot \neq [\![0]\!]$

By the $\begin{cases}\text{adequacy property}\\ \text{soundness property}\end{cases}$ we have $\begin{cases} F\,0\,\Omega \Downarrow_{nat} 0 \\ G\,0\,\Omega \not\Downarrow_{nat} 0\end{cases}$

so, as before, $F \not\cong_{ctx} G : nat \to nat \to nat$

2009 Paper 9 Question 6

(*b*) (*i*) Define a closed PCF term $H : (nat \to nat \to nat) \to nat \to nat \to nat$ such that $[\![\mathbf{fix}(H)]\!] \in \big(\mathbb{N}_\perp \to (\mathbb{N}_\perp \to \mathbb{N}_\perp)\big)$ satisfies

$$[\![\mathbf{fix}(H)]\!]\,(i)\,(j) \;=\; \max(i, j)$$

for all $i, j \in \mathbb{N}$. [4 marks]

IDEA $\max(i,j)$ is uniquely specified by

$$\begin{cases} \max(0,j) = j \\ \max(i,0) = i \\ \max(i+1,j+1) = \max(i,j)+1 \end{cases}$$

IDEA $\max(i,j)$ is uniquely specified by

$$\begin{cases} \max(0,j) = j \\ \max(i,0) = i \\ \max(i+1,j+1) = \max(i,j)+1 \end{cases}$$

So $\max(i,j) = [\![\mathrm{fix}(H)]\!](i)(j)$ for

$$\boxed{\begin{array}{l} H \overset{\text{def}}{=} \mathbf{fn}\ f : nat \to nat \to nat .\ \mathbf{fn}\ x : nat .\ \mathbf{fn}\ y : nat . \\ \qquad \mathbf{if}\ zero(x)\ \mathbf{then}\ y\ \mathbf{else} \\ \qquad \mathbf{if}\ zero(y)\ \mathbf{then}\ x\ \mathbf{else} \\ \qquad succ(f\,(pred(x))(pred(y))) \end{array}}$$

2009 Paper 9 Question 6

(ii) Let

$$S \stackrel{\text{def}}{=} \{\, f \in \big(\mathbb{N}_\perp \to (\mathbb{N}_\perp \to \mathbb{N}_\perp)\big) \mid f(x)(y) = f(y)(x) \text{ for all } x, y \in \mathbb{N}_\perp \,\}$$

Show that the subset $S \subseteq \big(\mathbb{N}_\perp \to (\mathbb{N}_\perp \to \mathbb{N}_\perp)\big)$ is admissible. [4 marks]

(iii) Show that

$$[\![\mathbf{fix}(H)]\!]\,(x)\,(y) \;=\; [\![\mathbf{fix}(H)]\!]\,(y)\,(x)$$

for all $x, y \in \mathbb{N}_\perp$. [5 marks]

[Hint: Use Scott's Fixed-Point Induction Principle.]

Chain-closed and admissible subsets

Let D be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D

$$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \left(\bigsqcup_{n \geq 0} d_n\right) \in S$$

If D is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of D and $\perp \in S$.

$$\perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp}(x) = \perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp}$$

$$\perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp}(x)(y) = \perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp}(y) = \perp_{\mathbb{N}_\perp}$$

Similarly $\perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp}(y)(x) = \perp_{\mathbb{N}_\perp}$

So $\perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp} \in S$

$$\perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp}(x)(y) = \perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp}(y) = \perp_{\mathbb{N}_\perp}$$

Similarly $\perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp}(y)(x) = \perp_{\mathbb{N}_\perp}$

So $\perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp} \in S$

If $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \cdots$ in $\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp$

and $f_n \in S$ for all $n = 0, 1, 2, \ldots$, then

$$\left(\bigsqcup_{n \geq 0} f_n\right)(x)(y) = \left(\bigsqcup_{n \geq 0} f_n(x)\right)(y)$$

$$\perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp}(x)(y) = \perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp}(y) = \perp_{\mathbb{N}_\perp}$$

Similarly $\perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp}(y)(x) = \perp_{\mathbb{N}_\perp}$

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and $f_n \in S$ for all $n = 0, 1, 2, \ldots$, then

$$\Big(\bigsqcup_{n \geq 0} f_n\Big)(x)(y) = \Big(\bigsqcup_{n \geq 0} f_n(x)\Big)(y) = \bigsqcup_{n \geq 0} f_n(x)(y)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\quad f_n \in S}{\shortparallel}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \bigsqcup_{n \geq 0} f_n(y)(x)$$

$$\perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp}(x)(y) = \perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp}(y) = \perp_{\mathbb{N}_\perp}$$

Similarly $\perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp}(y)(x) = \perp_{\mathbb{N}_\perp}$

So $\perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp} \in S$

If $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \cdots$ in $\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp$

and $f_n \in S$ for all $n = 0, 1, 2, \ldots$, then

$$\left(\bigsqcup_{n \geq 0} f_n\right)(x)(y) = \left(\bigsqcup_{n \geq 0} f_n(x)\right)(y) = \bigsqcup_{n \geq 0} f_n(x)(y)$$

$$\parallel \qquad f_n \in S$$

$$\left(\bigsqcup_{n \geq 0} f_n\right)(y)(x) = \cdots = \bigsqcup_{n \geq 0} f_n(y)(x)$$

$$\perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp}(x)(y) = \perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp}(y) = \perp_{\mathbb{N}_\perp}$$

Similarly $\perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp}(y)(x) = \perp_{\mathbb{N}_\perp}$

So $\perp_{\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp} \in S$

If $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ in $\mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp$

and $f_n \in S$ for all $n = 0, 1, 2, \ldots$, then

$$(\textstyle\bigsqcup_{n\geq 0} f_n)(x)(y) = (\textstyle\bigsqcup_{n\geq 0} f_n)(y)(x)$$

so $\bigsqcup_{n\geq 0} f_n \in S$

2009 Paper 9 Question 6

(ii) Let

$$S \stackrel{\text{def}}{=} \{\, f \in \big(\mathbb{N}_\perp \to (\mathbb{N}_\perp \to \mathbb{N}_\perp)\big) \mid f(x)(y) = f(y)(x) \text{ for all } x, y \in \mathbb{N}_\perp \,\}$$

Show that the subset $S \subseteq \big(\mathbb{N}_\perp \to (\mathbb{N}_\perp \to \mathbb{N}_\perp)\big)$ is admissible. [4 marks]

(iii) Show that

$$[\![\mathbf{fix}(H)]\!]\,(x)\,(y) \;=\; [\![\mathbf{fix}(H)]\!]\,(y)\,(x)$$

for all $x, y \in \mathbb{N}_\perp$. [5 marks]

[Hint: Use Scott's Fixed-Point Induction Principle.]

Scott's Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain D.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of f is in S, *i.e.* that

$$\mathit{fix}(f) \in S \ ,$$

it suffices to prove

$$\forall d \in D \ (d \in S \ \Rightarrow \ f(d) \in S) \ .$$

$$\boxed{\begin{array}{l} H \overset{\text{def}}{=} \mathbf{fn}\ f : nat \to nat \to nat\ .\ \mathbf{fn}\ x : nat\ .\ \mathbf{fn}\ y : nat\ . \\ \qquad \mathbf{if}\ zero(x)\ \mathbf{then}\ y\ \mathbf{else} \\ \qquad \mathbf{if}\ zero(y)\ \mathbf{then}\ x\ \mathbf{else} \\ \qquad succ(f\,(pred(x))(pred(y))) \end{array}}$$

Want to show $[\![\mathbf{fix}(H)]\!] \in S$

i.e. $fix([\![H]\!]) \in S$

Since S is admissible, by Scott Induction suffices to show

$$\forall f \in \mathbb{N}_\perp \to \mathbb{N}_\perp \to \mathbb{N}_\perp .\ f \in S \Rightarrow [\![H]\!](f) \in S$$

$$H \stackrel{\text{def}}{=} \mathbf{fn}\ f : nat \to nat \to nat .\ \mathbf{fn}\ x : nat .\ \mathbf{fn}\ y : nat .$$
$$\quad \mathbf{if}\ zero(x)\ \mathbf{then}\ y\ \mathbf{else}$$
$$\quad \mathbf{if}\ zero(y)\ \mathbf{then}\ x\ \mathbf{else}$$
$$\quad succ(f\,(pred(x))(pred(y)))$$

Thus

$$[\![H]\!](f)(x)(y) = \begin{cases} \perp & \text{if } x = \perp \\ y & \text{if } x = 0 \\ \perp & \text{if } x > 0\ \&\ y = \perp \\ x & \text{if } x > 0\ \&\ y = 0 \\ f(x-1)(y-1)+1 & \text{if } x > 0\ \&\ y > 0 \end{cases}$$

$$[\![H]\!](f)(y)(x) = \begin{cases} \perp & \text{if } y = \perp \\ x & \text{if } y = 0 \\ \perp & \text{if } y > 0 \;\&\; x = \perp \\ y & \text{if } y > 0 \;\&\; x = 0 \\ f(y-1)(x-1)+1 & \text{if } y > 0 \;\&\; x > 0 \end{cases}$$

$$[\![H]\!](f)(x)(y) = \begin{cases} \perp & \text{if } x = \perp \\ y & \text{if } x = 0 \\ \perp & \text{if } x > 0 \;\&\; y = \perp \\ x & \text{if } x > 0 \;\&\; y = 0 \\ f(x-1)(y-1)+1 & \text{if } x > 0 \;\&\; y > 0 \end{cases}$$

$$[\![H]\!](f)(y)(x) = \begin{cases} \bot & \text{if } y = \bot \text{ or } x = \bot \\ \max(y, x) & \text{if } y = 0 \text{ or } x = 0 \\ f(y-1)(x-1)+1 & \text{if } y > 0 \;\&\; x > 0 \end{cases}$$

$$[\![H]\!](f)(x)(y) = \begin{cases} \bot & \text{if } x = \bot \text{ or } y = \bot \\ \max(x, y) & \text{if } x = 0 \text{ or } y = 0 \\ f(x-1)(y-1)+1 & \text{if } x > 0 \;\&\; y > 0 \end{cases}$$

$$[\![H]\!](f)(y)(x) = \begin{cases} \perp & \text{if } y=\perp \text{ or } x=\perp \\ \max(y,x) & \text{if } y=0 \text{ or } x=0 \\ f(y-1)(x-1)+1 & \text{if } y>0 \;\&\; x>0 \end{cases}$$

$\|$ provided $f \in S$

$$[\![H]\!](f)(x)(y) = \begin{cases} \perp & \text{if } x=\perp \text{ or } y=\perp \\ \max(x,y) & \text{if } x=0 \text{ or } y=0 \\ f(x-1)(y-1)+1 & \text{if } x>0 \;\&\; y>0 \end{cases}$$

i.e. $[\![H]\!](f) \in S$ when $f \in S$