Chapter 4 [p 45]

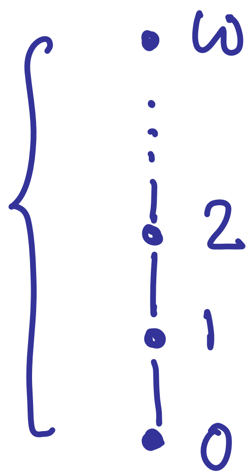
Scott Induction

Chain-closed and admissible subsets

Let D be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D

$$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \left(\bigsqcup_{n \geq 0} d_n\right) \in S$$

If D is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of D and $\perp \in S$.

For the domain $\Omega =$ 

the subset

- $\{2,4,6\}$ is chain-closed, <u>not</u> admissible
- $\{0,2,4,6\}$ is (chain-closed &) admissible
- $\{0,2,4,6,\dots\}$ is <u>not</u> chain-closed
- $\{0,2,4,6,\dots\} \cup \{\omega\}$ is admissible

Chain-closed and admissible subsets

Let D be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D

$$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \Big(\bigsqcup_{n \geq 0} d_n\Big) \in S$$

If D is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of D and $\perp \in S$.

A property $\Phi(d)$ of elements $d \in D$ is called *chain-closed* (resp. *admissible*) iff $\{d \in D \mid \Phi(d)\}$ is a *chain-closed* (resp. *admissible*) subset of D.

Scott's Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain D.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of f is in S, *i.e.* that

$$\mathit{fix}(f) \in S \ ,$$

it suffices to prove

$$\forall d \in D \ (d \in S \ \Rightarrow \ f(d) \in S) \ .$$

Tarski's Fixed Point Theorem

Let $f : D \to D$ be a continuous function on a domain D. Then

- f possesses a least pre-fixed point, given by

$$fix(f) = \bigsqcup_{n \geq 0} f^n(\perp).$$

- Moreover, $fix(f)$ is a fixed point of f, *i.e.* satisfies $f(fix(f)) = fix(f)$, and hence is the least fixed point of f.

where $\begin{cases} f^0(\perp) \triangleq \perp \\ f^{n+1}(\perp) \triangleq f(f^n(\perp)) \end{cases}$

Proof of the Scott Induction Principle

If we know $\forall d \in D.\ d \in S \Rightarrow f(d) \in S$, then

$\perp \in S$ since S is admissible

Proof of the Scott Induction Principle

If we know $\boxed{\forall d \in D.\ \ d \in S \Rightarrow f(d) \in S}$, then

$\bot \in S$ since S is admissible

so $f(\bot) \in S$ by

Proof of the Scott Induction Principle

If we know $\boxed{\forall d \in D.\ \ d \in S \Rightarrow f(d) \in S}$, then

$\perp \in S$ since S is admissible

so $f(\perp) \in S$

so $f(f(\perp)) \in S$ by

....

$f^n(\perp) \in S$ for all $n \in \mathbb{N}$

Proof of the Scott Induction Principle

If we know $\forall d \in D.\ d \in S \Rightarrow f(d) \in S$, then

$\perp \in S$ since S is admissible

so $f(\perp) \in S$

so $f(f(\perp)) \in S$

....

$f^n(\perp) \in S$ for all $n \in \mathbb{N}$

Hence $\bigsqcup_{n \geq 0} f^n(\perp) \in S$ since S is admissible

that is, $\boxed{fix(f) \in S}$

Q.E.D.

Example 4.2.1

Given $\begin{cases} \text{domain } D \\ \text{continuous function } f : D \times D \times D \to D \end{cases}$

then $\begin{cases} g : D \times D \to D \times D \\ g(d_1, d_2) = (f(d_1, d_1, d_2), f(d_1, d_2, d_2)) \end{cases}$ is continuous.

So by Tarski's FPT we get $fix(g) \in D \times D$.

$\boxed{\text{Claim: } u_1 = u_2, \text{ where } (u_1, u_2) = fix(g)}$

Proof : by Scott Induction...

Example 4.2.1

$$\begin{cases} g : D \times D \to D \times D \\ g(d_1, d_2) = (f(d_1, d_1, d_2), f(d_1, d_2, d_2)) \end{cases}$$

$$\boxed{\text{Claim: } u_1 = u_2, \text{ where } (u_1, u_2) = fix(g)}$$

<u>Proof</u> $\Delta \triangleq \{(d, d) \mid d \in D\}$ is an admissible subset of $D \times D$ because

- $(\bot, \bot) \in \Delta$
- $(d_0, d_0') \sqsubseteq (d_1, d_1') \sqsubseteq \cdots$ & $\forall n. (d_n, d_n') \in \Delta$ implies

  $\bigsqcup_{n \geq 0} (d_n, d_n') = (\bigsqcup_{n \geq 0} d_n, \bigsqcup_{n \geq 0} d_n') = (\bigsqcup_{n \geq 0} d_n, \bigsqcup_{n \geq 0} d_n) \in \Delta$

Example 4.2.1

$$\begin{cases} g : D \times D \to D \times D \\ g(d_1, d_2) = (f(d_1, d_1, d_2), f(d_1, d_2, d_2)) \end{cases}$$

$$\boxed{\text{Claim: } u_1 = u_2, \text{ where } (u_1, u_2) = \operatorname{fix}(g)}$$

<u>Proof</u> $\Delta = \{(d,d) \mid d \in D\}$ admissible

and $\forall (d,d') \in D \times D.\ (d,d') \in \Delta \Rightarrow g(d,d') \in \Delta$

because

$$(d,d') \in \Delta \Rightarrow d = d'$$

$$\Rightarrow g(d,d') = (f(d,d,d), f(d,d,d)) \in \Delta$$

Example 4.2.1

$$\begin{cases} g : D \times D \to D \times D \\ g(d_1, d_2) = (f(d_1, d_1, d_2), f(d_1, d_2, d_2)) \end{cases}$$

$\boxed{\text{Claim: } u_1 = u_2 \text{, where } (u_1, u_2) = fix(g)}$

<u>Proof</u> $\Delta = \{(d,d) \mid d \in D\}$ admissible

and $\forall (d,d') \in D \times D.\ (d,d') \in \Delta \Rightarrow g(d,d') \in \Delta$

So by Scott Induction

$fix(g) \in \Delta$

Q.E.D.

Building chain-closed subsets (I)

Let D, E be cpos.

Basic relations:

- For every $d \in D$, the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{ x \in D \mid x \sqsubseteq d \}$$

  of D is chain-closed.

Building chain-closed subsets (I)

Let D, E be cpos.

Basic relations:

- For every $d \in D$, the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{\, x \in D \mid x \sqsubseteq d \,\}$$

  of D is chain-closed.

- The subsets

$$\{(x, y) \in D \times D \mid x \sqsubseteq y\}$$

  and

$$\{(x, y) \in D \times D \mid x = y\}$$

  of $D \times D$ are chain-closed.

Building chain-closed subsets (II)

Inverse image:

Let $f : D \to E$ be a continuous function.

If S is a chain-closed subset of E then the inverse image

$$f^{-1}S \;=\; \{x \in D \mid f(x) \in S\}$$

is a chain-closed subset of D.

Building chain-closed subsets (II)

Inverse image:

Let $f : D \to E$ be a continuous function.

If S is a chain-closed subset of E then the inverse image

$$f^{-1}S = \{x \in D \mid f(x) \in S\}$$

is a chain-closed subset of D.

Proof: if $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D & $\forall n.\ d_n \in f^{-1}S$

Building chain-closed subsets (II)

Inverse image:

Let $f : D \to E$ be a continuous function.

If S is a chain-closed subset of E then the inverse image

$$f^{-1}S = \{x \in D \mid f(x) \in S\}$$

is a chain-closed subset of D.

Proof: if $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D & $\forall n.\ d_n \in f^{-1}S$

then $\forall n.\ f(d_n) \in S$, so $\bigsqcup_{n \geq 0} f(d_n) \in S$ ('cos S ch.-cl.)

Building chain-closed subsets (II)

Inverse image:

Let $f : D \to E$ be a continuous function.

If S is a chain-closed subset of E then the inverse image

$$f^{-1}S = \{x \in D \mid f(x) \in S\}$$

is a chain-closed subset of D.

Proof: if $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D & $\forall n.\ d_n \in f^{-1}S$

then $\forall n.\ f(d_n) \in S$, so $\bigsqcup_{n \geq 0} f(d_n) \in S$ ('cos S ch.-cl.)

so $f(\bigsqcup_{n \geq 0} d_n) \in S$ ('cos f cts.)

so $\bigsqcup_{n \geq 0} d_n \in f^{-1}S$

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\perp) \sqsubseteq g(\perp) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\perp) \sqsubseteq g(\perp) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

Proof by Scott induction.

Consider the admissible property $\Phi(x) \equiv \big(f(x) \sqsubseteq g(x)\big)$ of D.

Since

$$f(x) \sqsubseteq g(x) \Rightarrow g(f(x)) \sqsubseteq g(g(x)) \Rightarrow f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\mathit{fix}(g)) \sqsubseteq g(\mathit{fix}(g)) \ .$$

Given continuous functions

$$f : D \to E \qquad g : D \to F$$

we get a continuous function

$$\langle f, g \rangle : D \to E \times F$$

given by $\boxed{\langle f, g \rangle = \lambda d \in D.\ (f(d), g(d))}$

(Check:

$$\begin{aligned}
\langle f, g \rangle \left(\textstyle\bigsqcup_{n \geq 0} d_n \right) &= \left(f\left(\textstyle\bigsqcup_{n \geq 0} d_n \right), g\left(\textstyle\bigsqcup_{n \geq 0} d_n \right) \right) \\
&= \left(\textstyle\bigsqcup_{n \geq 0} f(d_n), \textstyle\bigsqcup_{n \geq 0} g(d_n) \right) \\
&= \textstyle\bigsqcup_{n \geq 0} (f(d_n), g(d_n)) \\
&= \textstyle\bigsqcup_{n \geq 0} \langle f, g \rangle (d_n)
\end{aligned}$$

)

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

Proof by Scott induction.

Consider the admissible property $\Phi(x) \equiv \big(f(x) \sqsubseteq g(x)\big)$ of D.

$\{x \mid \Phi(x)\} = \langle f, g \rangle^{-1} \sqsubseteq$

Since

$$f(x) \sqsubseteq g(x) \Rightarrow g(f(x)) \sqsubseteq g(g(x)) \Rightarrow f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\mathit{fix}(g)) \sqsubseteq g(\mathit{fix}(g)) \ .$$

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

Proof by Scott induction.

Consider the admissible property $\Phi(x) \equiv \big(f(x) \sqsubseteq g(x)\big)$ of D.

Since

$$f(x) \sqsubseteq g(x) \Rightarrow g(f(x)) \sqsubseteq g(g(x)) \Rightarrow f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\mathit{fix}(g)) \sqsubseteq g(\mathit{fix}(g)) \sqsubseteq \mathit{fix}(g) \text{ by (lfp1) for } g$$

so by (lfp2) for f, we have

$\mathit{fix}(f) \sqsubseteq \mathit{fix}(g)$ Q.E.D.

Building chain-closed subsets (III)

Logical operations:

- If $S, T \subseteq D$ are chain-closed subsets of D then

  $$S \cup T \qquad \text{and} \qquad S \cap T$$

  are chain-closed subsets of D.

- If $\{ S_i \}_{i \in I}$ is a family of chain-closed subsets of D indexed by a set I, then $\bigcap_{i \in I} S_i$ is a chain-closed subset of D.

- If a property $P(x, y)$ determines a chain-closed subset of $D \times E$, then the property $\forall x \in D.\ P(x, y)$ determines a chain-closed subset of E.

S, T chain-closed $\Rightarrow$ $S \cup T$ chain-closed

Suppose $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots$ in D & $\forall n.\ d_n \in S \cup T$

If $\bigsqcup_{n \geq 0} d_n \in S$, we are done.

So suppose $\bigsqcup_{n \geq 0} d_n \notin S$

For each $m \geq 0$,

$$(\forall n \geq m.\ d_n \in S) \Rightarrow \bigsqcup_{n \geq 0} d_n = \bigsqcup_{n \geq m} d_n \in S \quad \text{※}$$

S, T chain-closed $\Rightarrow S \cup T$ chain-closed

Suppose $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots$ in D & $\forall n.\ d_n \in S \cup T$

If $\bigsqcup_{n \geq 0} d_n \in S$, we are done.

So suppose $\bigsqcup_{n \geq 0} d_n \notin S$

For each $m \geq 0$,

$$(\forall n \geq m.\ d_n \in S) \Rightarrow \bigsqcup_{n \geq 0} d_n = \bigsqcup_{n \geq m} d_n \in S$$

So $\neg(\forall n \geq m.\ d_n \in S)$

i.e. $\exists n \geq m.\ d_n \in T$ since

S, T chain-closed $\Rightarrow$ $S \cup T$ chain-closed

Suppose $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots$ in D & $\forall n.\ d_n \in S \cup T$

If $\bigsqcup_{n \geq 0} d_n \in S$, we are done.

So suppose $\bigsqcup_{n \geq 0} d_n \notin S$

For each $m \geq 0$, $\exists\, n \geq m .\ d_n \in T$

S, T chain-closed $\Rightarrow S \cup T$ chain-closed

Suppose $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots$ in D & $\forall n.\ d_n \in S \cup T$

If $\bigsqcup_{n \geq 0} d_n \in S$, we are done.

So suppose $\bigsqcup_{n \geq 0} d_n \notin S$

For each $m \geq 0$, $\exists n \geq m.\ d_n \in T$

So we can choose $n_0 \leq n_1 \leq n_2 \leq \cdots$ satisfying
$\forall m.\ \ m \leq n_m$ & $d_{n_m} \in T$.

So $\bigsqcup_{n \geq 0} d_n = \bigsqcup_{m \geq 0} d_{n_m} \in T$

Q.E.D.

Building chain-closed subsets (III)

Logical operations:

- If $S, T \subseteq D$ are chain-closed subsets of D then

$$S \cup T \qquad \text{and} \qquad S \cap T$$

  are chain-closed subsets of D.

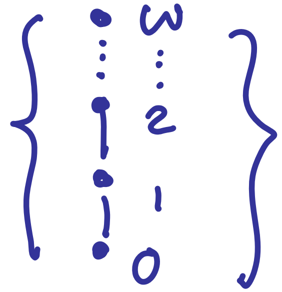
- If $\{ S_i \}_{i \in I}$ is a family of chain-closed subsets of D indexed by a set I, then $\bigcap_{i \in I} S_i$ is a chain-closed subset of D.

  $= \{ d \in D \mid \forall i.\ d \in S_i \}$

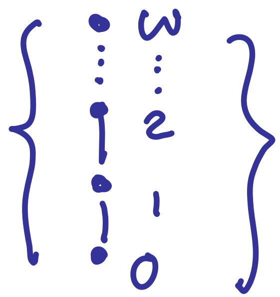
- If a property $P(x, y)$ determines a chain-closed subset of $D \times E$, then the property $\forall x \in D.\ P(x, y)$ determines a chain-closed subset of E.

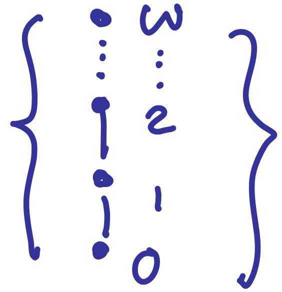
N.B. in general $\bigcup_{i \in I} S_i = \{ d \mid \exists i.\ d \in S_i \}$ & $D - S$ need not be chain-closed.

for each $i \in \mathbb{N}$

$S_i = \{0, 1, 2, \dots, i\}$ is chain-closed subset of the domain $\Omega =$ 

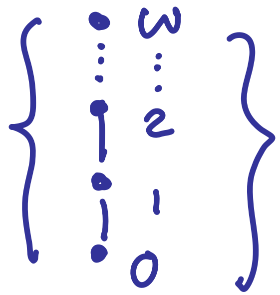
but

$\bigcup_{i \in \mathbb{N}} S_i = \mathbb{N}$ is not a chain-closed subset of the domain $\Omega =$ 

$S = \{0, 2, 4, \ldots\} \cup \{\omega\}$ is chain-closed subset of the domain $\Omega =$ 

but

$D - S = \{1, 3, 5, \ldots\}$

is not a chain-closed subset of the domain $\Omega =$ 

Example (III): Partial correctness

Let $\mathcal{F} : State \rightharpoonup State$ be the denotation of

$$\textbf{while } X > 0 \textbf{ do } (Y := X * Y; X := X - 1) \ .$$

For all $x, y \geq 0$,

$$\mathcal{F}[X \mapsto x, Y \mapsto y] \downarrow$$
$$\implies \mathcal{F}[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y].$$

Recall that

$$\mathcal{F} = \mathit{fix}(f)$$

where $f : (State \rightharpoonup State) \to (State \rightharpoonup State)$ is given by

$$f(w) = \lambda(x, y) \in State. \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$

Proof by Scott induction.

We consider the admissible subset of $(State \rightharpoonup State)$ given by

$$S = \left\{ w \;\middle|\; \begin{array}{l} \forall x, y \geq 0. \\ \quad w[X \mapsto x, Y \mapsto y]\downarrow \\ \qquad \Rightarrow w[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y] \end{array} \right\}$$

and show that

$$w \in S \implies f(w) \in S \ .$$

Suppose $w \in S$. Want to show $f(w) \in S$, i.e.

$$x, y \geq 0 \ \& \ f(w)(x, y)\downarrow \implies f(w)(x,y) = (0, !x \cdot y)$$

So suppose $x, y \geq 0 \ \& \ f(w)(x,y)\downarrow$

Case $x = 0$:

$$f(w)(x,y) \underset{\substack{\text{by def.}\\ \text{of } f}}{=} (x,y) \underset{\substack{\text{Since}\\ x=0}}{=} (0,y) \underset{\substack{\text{Since}\\ 0!=1}}{=} (0, !0 \cdot y) \underset{\substack{\text{since}\\ x=0}}{=} (0, !x \cdot y) \quad \checkmark$$

Suppose $w \in S$. Want to show $f(w) \in S$, i.e.

$$x, y \geq 0 \ \& \ f(w)(x,y)\downarrow \implies f(w)(x,y) = (0, !x \cdot y)$$

So suppose $x, y \geq 0$ & $f(w)(x,y)\downarrow$

Case $x > 0$: Since $f(w)(x,y)\downarrow$

get $w(x-1, x \cdot y)\downarrow$ by definition of f

Suppose $w \in S$. Want to show $f(w) \in S$, i.e.

$x, y \geq 0$ & $f(w)(x, y)\downarrow \Rightarrow f(w)(x,y) = (0, !x \cdot y)$

So suppose $x, y \geq 0$ & $f(w)(x,y)\downarrow$

Case $x > 0$: Since $f(w)(x,y)\downarrow$

get $w(x-1, x \cdot y)\downarrow$ by definition of f

But $x-1, x \cdot y \geq 0$ & $w \in S$

so $w(x-1, x \cdot y) = (0, !(x-1) \cdot (x \cdot y))$ by def^n of S

Suppose $w \in S$. Want to show $f(w) \in S$, i.e.

$$x, y \geq 0 \;\&\; f(w)(x,y)\downarrow \Rightarrow f(w)(x,y) = (0, !x \cdot y)$$

So suppose $x, y \geq 0$ & $f(w)(x,y)\downarrow$

Case $x > 0$: Since $f(w)(x,y)\downarrow$

get $w(x-1, x \cdot y)\downarrow$ by definition of f

But $x-1, x \cdot y \geq 0$ & $w \in S$

so $w(x-1, x \cdot y) = (0, !(x-1) \cdot (x \cdot y))$ by def.ⁿ of S

$$= (0, !x \cdot y)$$

so $f(w)(x \cdot y) = w(x-1, x \cdot y) = (0, !x \cdot y)$ ✓

def.ⁿ of f